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Decomposition of Balanced Matrices.
Part V: Goggles

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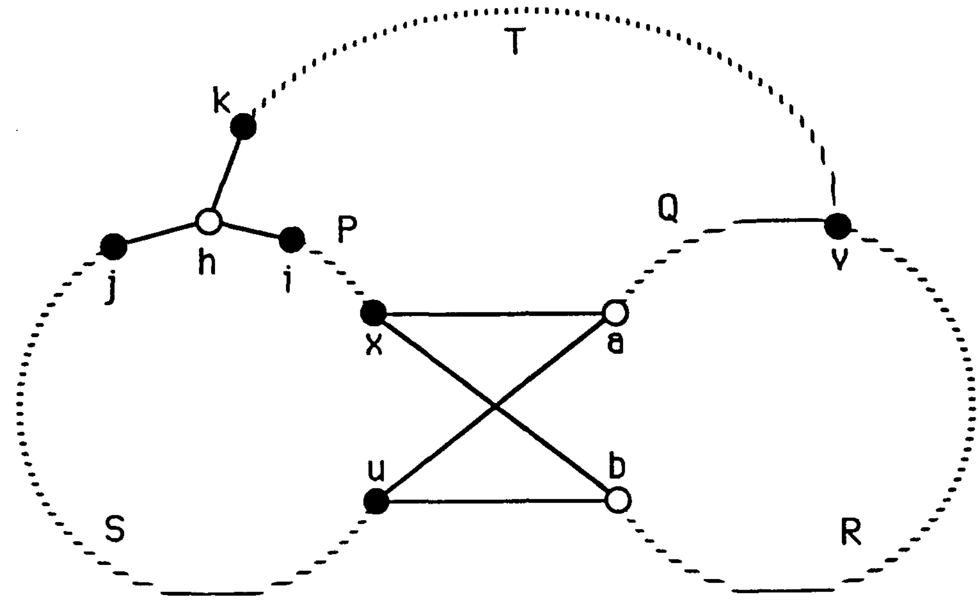


Figure 1: Goggles

1 Introduction

In this part we assume that G ~~is~~ a bipartite graph that is signable to be balanced and contains goggles $\Gamma = Go(P,Q,R,S,T)$, see Figure 1. We use the following notation: $P = x,\ldots,i,h$, $Q = a,\ldots,v$, $R = b,\ldots,v$, $S = u,\ldots,j,h$ and $T = h,k,\ldots,v$.

The paths P,Q,R,S have length greater than 1, but the path T may be of length 1 in which case $h \in N(v)$ and $k = v$. Assume w.l.o.g. that $a \in V^r$. Then $x,u,v \in V^c$ and $b,h \in V^r$. Further, we assume that G does not contain

- a wheel,
- connected squares,
- a connected 6-hole,
- an R_{10} configuration,
- an extended star cutset.

Since G does not contain an extended star cutset, it follows from Part III that G does not contain

- a parachute with long sides and long top,
- a parachute with long sides, short top and short middle.

The main result of this paper is that G has a 2-join. This requires an understanding of the possible paths connecting nodes of Γ. But in order to identify these paths, we need to first study the structure of the strongly adjacent nodes to Γ. This is done in the next section. In Section 3 we identify the bicliques formed by the nodes of Γ and strongly adjacent nodes. In Section 4 we study the direct connections between nodes in Γ. In Sections 5 to 9 we study the structure of direct connections to Γ from strongly adjacent nodes and neighbors of node h not in Γ. Finally, in Section 10 we prove the 2-join theorem.

We use the following results about parachutes:

Corollary 1.1 *If Π is a parachute with long sides, long middle, short top v_1, t, v_2 and center node v, then there exists a direct connection of Type d[3.3(III)] or d1[4.1(III)], connecting the bottom of Π to the top $\{t\}$. Furthermore, every direct connection between the bottom and the top of Π avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$ is of Type d[3.3(III)] or d1[4.1(III)].*

Proof: The result is a consequence of Theorem 8.1(III). □

Lemma 1.2 *Let Π be a parachute with long sides, long middle and short top v_1, t, v_2 and center node v. There does not exist a chordless path $x_1, \dots, x_m$ such that*

(i) $m \geq 2$ *and* $x_p \in V(G) \setminus V(\Pi)$, $1 \leq p \leq m$,

(ii) *Node x_1 is adjacent to one of the nodes v_1, v_2 or v, and to no other node of Π.*

(iii) *Node x_m is adjacent to t and to no other node of Π.*

(iv) *For $2 \leq p \leq m-1$, node x_p has no neighbor in Π.*

Proof: Assume such a path exists. Since G has no extended star cutset, there exists a direct connection $Y = y_1, \dots, y_n$ between the bottom of Π and $\{x_1, \dots, x_m\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2))$. This implies a direct connection $W = y_1 = w_1, \ldots, w_l$ from the bottom of Π to $\{t\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$. By Corollary 1.1, the direct connection W is of Type d[3.3(III)] or d1[4.1(III)].

If W is of Type d[3.3(III)], let $s \in V(M)$ be the neighbor of w_1. Since s is not adjacent to v, there is a $3PC(v, s)$ irrespective of whether Y contains neighbors of v_1, v_2 or t.

If W is of Type d1[4.1(III)], there is a $3PC(v, y_1)$. □

Lemma 1.3 *Let Π be a parachute with long sides, long middle and short top v_1, t, v_2, center node v and bottom node z. There exists no chordless path $x_1, \ldots, x_m$ such that*

(i) $m \geq 2$ *and* $x_p \in V(G) \setminus V(\Pi)$, $1 \leq p \leq m$,

(ii) *Node x_1 is adjacent to either v_1 or v_2, say v_1, and to no other node of Π.*

(iii) *Node x_m is adjacent to v and to no other node of Π.*

(iv) *For $2 \leq p \leq m-1$, node x_p has no neighbor in Π.*

Proof: By Corollary 1.1, there exists a direct connection $Y = y_1, \ldots, y_n$ of Type d[3.3(III)] or d1[4.1(III)] between the bottom of Π and $\{t\}$. Let $s \in V(M)$ be the neighbor of y_1 closest to z. No node of Y is coincident with or adjacent to at least one of the nodes $x_1, \ldots, x_m$, for otherwise there is a $3PC(v, s)$ or $3PC(v, y_1)$. Consider the parachute with top path v_1, v, v_2, same side paths as Π and middle path $t, y_n, \ldots, y_1, s, M_{sz}, z$. Now the result follows by Lemma 1.2. □

2 Strongly Adjacent Nodes

Theorem 2.1 *Let $w \in V(G) \setminus V(\Gamma)$ be strongly adjacent to Γ. Then one of the following holds:*

- *Node w has two neighbors in Γ and both of these neighbors are in the same path P, Q, R, S or T.*

- *Node w is of one of the following types, see Figure 2.*

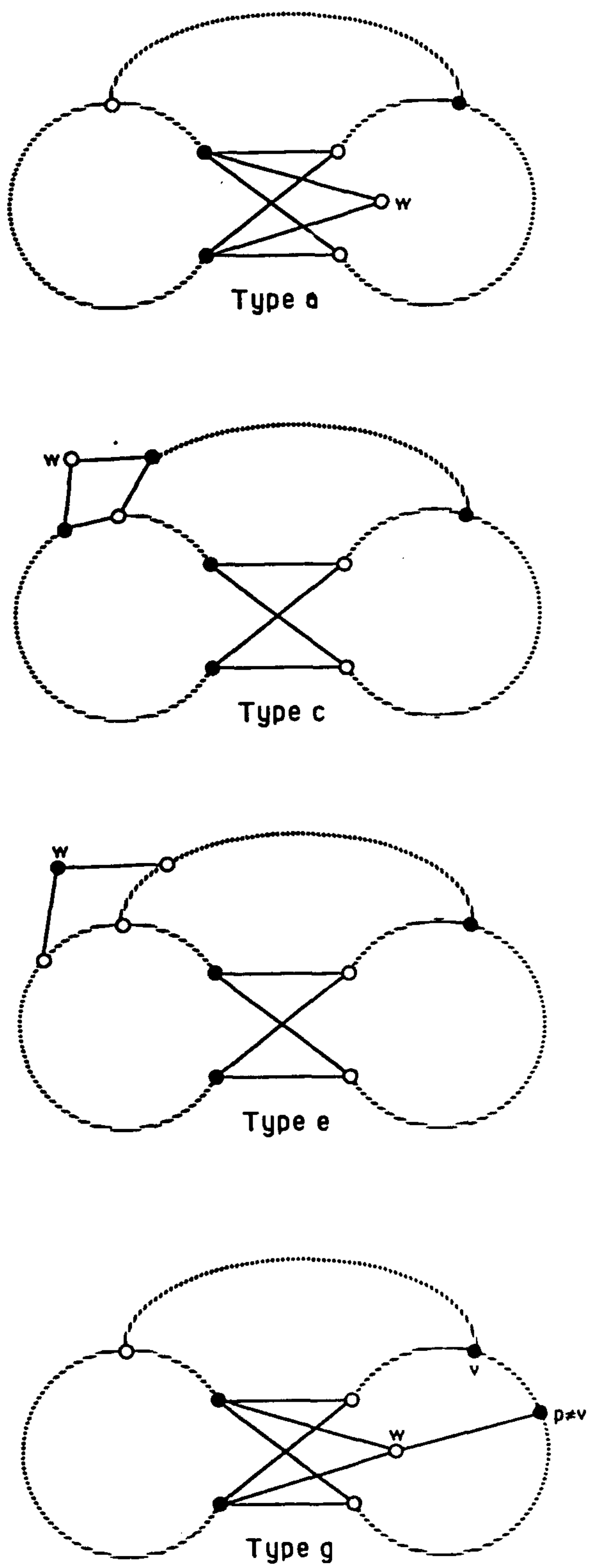


Figure 2: Strongly Adjacent Nodes

Type a *Node w has exactly two neighbors in Γ and w is adjacent to x and u (a and b resp.).*

Type b *Node w has exactly two neighbors in Γ and is adjacent to the two neighbors of h (v resp.) in P, S (Q, R resp.).*

Type c *Node w has exactly two neighbors in Γ and w is adjacent to the two neighbors of h (v resp.), one in T and one in either P or S (Q or R resp.).*

Type d *$w \in V^c$ ($w \in V^r$ resp.) has exactly two neighbors in Γ, one of them in $V(P) \setminus \{h\}$ and the other in $V(S) \setminus \{h\}$ ($V(Q) \setminus \{v\}$ and $V(R) \setminus \{v\}$ resp.).*

Type e *$w \in V^c$ ($w \in V^r$ resp.) has exactly two neighbors in Γ, one of them in $V(T) \setminus \{h, v\}$ and the other in $V(P) \cup V(S) \setminus \{h\}$ ($V(Q) \cup V(R) \setminus \{v\}$ resp.).*

Type f *Node w has three neighbors in Γ and w is adjacent to x, u (a, b resp.) and to a node of T which is not adjacent to h (v resp.).*

Type g *Node w has three neighbors in Γ and w is adjacent to x, u (a, b resp.) and to a node of $V(Q) \cup V(R) \setminus \{v\}$ ($V(P) \cup V(S) \setminus \{h\}$ resp.).*

Type h *Node w has three neighbors in Γ and w is adjacent to one node in T, to one node in $V(P) \setminus \{x\}$ and to one node in $V(S) \setminus \{u\}$ ($V(Q) \setminus \{a\}$ and $V(R) \setminus \{b\}$ resp.) and two of these three nodes are adjacent to h (v resp.).*

Proof: We consider first the case where w has two neighbors in Γ and then the case where w has three or more neighbors.

Case 1 Node w has two neighbors in Γ, say α and β.

If α and β belong to the same path P, Q, R, S or T, then w is as described in the first part of the theorem. Now assume α and β belong to distinct paths. Because of the symmetry between paths P, Q, R and S, we can assume w.l.o.g. that $\alpha \in V(P)$. We now have the following three subcases.

Case 1.1 $\beta \in V(S)$

Clearly, since α and β belong to distinct paths, $\alpha, \beta \neq h$. If $w \in V^c$, then w is of Type d. Suppose now $w \in V^r$. If $\alpha = x$, then $\beta = u$ for otherwise we have a $3PC(x, h)$. This yields a node w of Type a. Suppose now $\alpha \neq x$. By symmetry, $\beta \neq u$. Now, if α (β resp.) is not adjacent to h, there is a $3PC(h, \alpha)$ ($3PC(h, \beta)$ resp.). Hence both α and β are neighbors of h. This yields a node w of Type b.

Case 1.2 $\beta \in V(T)$

Clearly, $\alpha, \beta \neq h$. If $w \in V^c$, then w is of Type e. Suppose now $w \in V^r$. If α (β resp.) is not adjacent to h, there is $3PC(h, \alpha)$ ($3PC(h, \beta)$ resp.). Hence, both α and β are neighbors of h. This yields a node w of Type c.

Case 1.3 $\beta \in V(Q) \cup V(R)$

Because of symmetry, we can assume w.l.o.g. that $\beta \in V(Q)$ and $w \in V^c$. If β is not adjacent to v, there is a $3PC(v, \beta)$. Hence $\beta \neq a$. Now if $\alpha \neq h$ or if $|T| > 1$, there is a $3PC(x, \beta)$. Hence it follows that $\alpha = h$ and $\beta, h \in N(v)$. Then w is of Type c.

Case 2 Node w has three or more neighbors in Γ.

Clearly w has at most one neighbor in each of the paths P, Q, R, S, T, for otherwise there is a wheel. We now consider two cases depending upon whether $|N(w) \cap V(T)| = 0$ or 1.

Case 2.1 $|N(w) \cap V(T)| = 1$.

Now w has neighbors in at least two different paths P, Q, R, S. Because of symmetry, we assume w.l.o.g. that $|N(w) \cap V(P) \setminus \{h\}| = 1$. It follows that $h \notin N(w)$ and $|N(w) \cap (V(Q) \cup V(R)) \setminus \{v\}| = 0$ for otherwise there is a wheel. This implies that $|N(w) \cap V(S) \setminus \{h\}| = 1$ and w has exactly three neighbors in Γ. If $w \in V^c$, there is a $3PC(w, h)$. Hence $w \in V^r$. Let α, β, γ be the neighbors of w in P, S and T respectively. We now consider the following two subcases.

Case 2.1.1 α or β is a neighbor of h.

Assume w.l.o.g. that $\alpha \in N(h)$. If $\beta \in N(h)$, node w is of Type h. Suppose now $\beta \notin N(h)$. If $\gamma \notin N(h)$, there is a parachute with long top and long sides as follows. The top path is $\gamma, T_{\gamma h}, h, \alpha$, the side paths are $\alpha, P_{\alpha x}, x, a$ and $\gamma, T_{\gamma v}, v, Q_{va}, a$. The center node is w and the middle path is $w, \beta, S_{\beta u}, u, a$. Hence $\gamma \in N(h)$ and w is of Type h.

Case 2.1.2 Neither α nor β is a neighbor of h.

If $\alpha = x$ and $\beta = u$, it follows that $\gamma \notin N(h)$, for otherwise we have a parachute with long sides, short top and short middle as follows: The top path is x, b, u, the side paths are P and S, the center node is w and the middle path is w, γ, h. This yields a node w of Type f. Suppose now $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$. Now as in Case 2.1.1, we have a parachute with long top and long sides.

Case 2.2 $|N(w) \cap V(T)| = 0$.

Clearly, $h, v \notin N(w)$. Suppose w has four neighbors in Γ, one in each of the paths P, Q, R, S. Because of symmetry, we can assume w.l.o.g. that $w \in V^c$. This implies a $3PC(w,h)$. Consequently we can assume w.l.o.g. that $|N(w) \cap V(Q)| = 0$ and that w has exactly one neighbor in P, R and S. If $w \in V^c$, there is a $3PC(w,h)$. Hence $w \in V^r$. Let α, β and Γ be the neighbors of w in P, S and R respectively. If $\alpha = x$ and $\beta = u$, node w is of Type g. Suppose now $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$. If $\beta \notin N(h)$, there is a $3PC(a, \alpha)$. If $\beta \in N(h)$ and $\alpha \notin N(h)$, there is a $3PC(a, \beta)$. Hence $\alpha, \beta \in N(h)$. Now there are connected squares, which is a contradiction. $\square$

Theorem 2.2 *Among the goggles of G, let Γ be one with shortest top path T and, subject to this condition, with the fewest number of nodes. Let w be a strongly adjacent node to Γ. Then, one of the following holds:*

(i) *Node w is a twin of a node of Γ.*

(ii) *Node w is of Type a, b, c or d[2.1].*

Proof: If node w has two neighbors in one of the paths P, Q, R, S or T, then w must be a twin of one of the nodes of degree two in Γ, since otherwise Γ would not have shortest top path or would not have the fewest number of nodes, subject to this condition. If w is of Type e[2.1], there are goggles with a shorter top path. Similarly if w is of Type h[2.1] and is adjacent to the neighbors of h (v resp.) in P, S (Q, R resp.) but not to the neighbor of h (v resp.) in T. If w is of Type g[2.1] but is not a twin of a, b, u or x relative to Γ, there are goggles with a top path of the same length as T but with fewer nodes than Γ. Similarly if w is of Type h[2.1] and is adjacent to the neighbor of h (v resp.) in T and to exactly one of the neighbors of h (v resp.) in P, S (Q, R resp.). Finally, suppose w is of Type f[2.1]. Let γ be the neighbor of w in T. Now consider the parachute with side paths P and S, center node w, middle path $w, \gamma, T_{\gamma h}, h$ and top path x, b, u. Now, by Theorem 8.1(III), since G does not contain an extended star cutset, there exists a direct connection Y of Type d[3.3(III)] or d1[4.1(III)] between b and $T_{\gamma h} \setminus \{\gamma, h\}$. This parachute and the path Y induce goggles with a shorter top path than T. □

Throughout the rest of the paper, we assume that the goggles Γ has the shortest top path T and, subject to this, Γ has the fewest number of nodes. Therefore, Theorem 2.2 always applies.

3 Bicliques

Lemma 3.1 *Let y be a twin of node x and z a twin of node a. Then y and z are adjacent.*

Proof: Otherwise there is a $3PC(h, u)$. □

Lemma 3.2 *Let d be the node adjacent to x in P. If y is a twin of x, then y is adjacent to all the twins of d.*

Proof: Assume that y is not adjacent to a twin d^* of d. Now consider the goggles Γ^* obtained from Γ by replacing d with d^*. Let P^* be the new path $x, d^*, \ldots, h$. Apply Corollary 1.1 to the parachute Π^* with top path a, y, b, side paths Q and R, center node x and middle path x, P^*, h, T, v. Let Y be a path of Type d[3.3(III)] or d1[4.1(III)] relative to Π^*. In the goggles induced by the nodes of Y and of Π^*, node d violates Theorem 2.1. □

Lemma 3.3 *Let w be a Type a[2.1] strongly adjacent node adjacent to u and x. Then w is adjacent to all the twins of u and x relative to Γ.*

Proof: Assume that w is not adjacent to x^*, a twin of x. Apply Corollary 1.1 to the parachute with top path u, w, x, side paths P and S, center node b and middle path b, R, v, T, h. Let Y be a path of Type d[3.3(III)] or d1[4.1(III)]. Replacing a by w, Q by Y, and modifying R and T appropriately, we get goggles for which x^* violates Theorem 2.1, irrespective of whether x^* has neighbors in Y or not. □

Remark 3.4 *So nodes u, x, a, b, their twins and the Type a[2.1] nodes adjacent to u and x form a biclique.*

Similarly, nodes u, x, a, b, their twins and the Type a[2.1] nodes adjacent to a and b form a biclique.

Lemma 3.5 *There cannot exist nodes w and z of Type b[2.1] or Type c[2.1] having exactly one common neighbor in Γ.*

Proof: Otherwise there is an odd wheel with center h or v. □

Lemma 3.6 *Let w be a Type b[2.1] node adjacent to i and j. Then the top path T of the goggles is of length greater than 1 and w is adjacent to all the twins of i and j.*

Proof: Let Π_a be the parachute with top path i, w, j, side paths i, P_{ix}, x, a and j, S_{ju}, u, a, center node h and middle path h, T, v, Q, a. The parachute Π_b is defined similarly, replacing a by b and Q by R. Apply Corollary 1.1 to Π_a and Π_b and consider all resulting paths of Type d[3.3(III)] or d1[4.1(III)]. Let $Y = y_1, \ldots, y_n$ be a shortest such path. Assume w.l.o.g. that Y connects the bottom of Π_a to the top.

Assume now that the path T has length 1. Then y_1 is not adjacent to v but is adjacent to a node in Q. Furthermore, none of the nodes $y_2, \ldots, y_n$ is adjacent to R. If y_1 is adjacent to R, then y_1 must be of Type b or d[2.1]. Node y_1 is not of Type b[2.1] by definition of Y. Node y_1 is not of Type d[2.1] for otherwise, there is a $3PC(y_1, v)$. So, y_1 is not adjacent to R. Let s be the neighbor of y_1 which is closest to a in Q. It follows from the definition of Y that s is not a neighbor of v. This implies the existence of connected squares, with paths $P_1 = i, P_{ix}, x$; $P_2 = j, S_{ju}, u$; $P_3 = h, v, R, b$ and $P_4 = w, Y, s, Q_{sa}, a$. Hence the path T must have length greater than 1.

Assume that w is not adjacent to i^*, a twin of i. The parachute Π_a and the path Y induce goggles for which i^* violates Theorem 2.1, irrespective of whether i^* has neighbors in Y or not. □

Lemma 3.7 *Let w be a Type c[2.1] node adjacent to i and k. Then w is adjacent to all the twins of i and k relative to Γ.*

Proof: Apply Corollary 1.1 to the parachute with top path i, w, k, side paths i, P_{ix}, x, a and k, T_{kv}, v, Q, a, center node h and middle path h, S, u, a. Let Y be the resulting path of Type d[3.3(III)] or d1[4.1(III)]. In the goggles induced by the nodes of Y and of the above parachute, the twins of i and k must satisfy Theorem 2.1 and therefore they must be adjacent to w. □

Lemma 3.8 *The twins of h relative to Γ are adjacent to the twins of i, j, k.*

Proof: Suppose h^*, a twin of h is not adjacent to i^*, a twin of i. Now consider the goggles Γ^* obtained from Γ by replacing i with i^*. Let P^* be the new path $h, i^*, \ldots, x$. Node h^* is a Type c[2.1] node with respect to Γ^*. Apply Corollary 1.1 to the parachute with top path j, h^*, k, side paths j, S_{ju}, u, a and k, T_{kv}, v, Q, a, center node h and middle path h, P^*, x, a. Let Y be the resulting path of Type d[3.3(III)] or d1[4.1(III)]. In the goggles induced by the nodes of Y and of the above parachute, node i violates Theorem 2.1. Hence h^* must be adjacent i^*.

By symmetry, it follows that h^* is adjacent to all the twins of j.

Now, suppose h^* is not adjacent to k^*, a twin of k. Then, replacing k by k^* in Γ and using a similar argument as above, one can construct goggles in which node k violates Theorem 2.1. Hence h^* must be adjacent to k^*. □

Remark 3.9 *Lemmas 3.6 to 3.8 imply that nodes h, i, j, k, their twins and the nodes having two neighbors in the set $\{i, j, k\}$ form a biclique.*

Lemma 3.10 *Suppose $w \in V^c$ is a Type d[2.1] node adjacent to $p \in V(P)$ and $s \in V(S)$. Then w is adjacent to all the twins of p and s relative to Γ.*

Proof: Suppose w is not adjacent to p^*, a twin of p. Let p_1 and p_2 be the neighbors of p in P. Then there is a parachute with long side paths, short top path p_1, p^*, p_2 and short middle path p, w, s. □

Remark 3.11 *Let Γ^* be goggles obtained from Γ by replacing a node of Γ by one of its twins relative to Γ. Let U be the set consisting of nodes $u, x, a, b, h, v, i, j, k$ their twins and all Type a, b, c, d[2.1] nodes relative to Γ. Let U^* be defined accordingly, relative to Γ^*. By Lemmas 3.1, 3.2, 3.3, 3.6, 3.7, 3.8 and 3.10 the sets U and U^* coincide.*

Lemma 3.12 *Suppose the top path T of Γ is of length 1. Let w be a Type c[2.1] node adjacent to h and one of the neighbors of v in either Q or R. Let z be a Type c[2.1] node adjacent to v and to either i or j. Then w and z are adjacent.*

Proof: If w and z are not adjacent, there is a violation of Lemma 1.3, as follows. W.l.o.g. assume w is adjacent to the neighbor of v in Q, say t, and assume that z is adjacent to i. The parachute has top h, w, t, side paths S and t, Q_{ta}, a, u, center node v and middle path v, R, b, u. The extra path is h, i, z, v. □

Remark 3.13 *It follows that, when $|T| = 1$, the nodes h, v, their twins and the Type c[2.1] nodes form a biclique.*

4 Direct Connections from a Node in the Goggles

Let w be a node in the path P of Γ. Let W be the set of twins of w relative to Γ and let F be the set of edges of Γhaving w as endnode. In the partial graph $G \setminus F$, let $X = x_1, \ldots, x_n$ be a direct connection between w and $V(\Gamma) \setminus \{w\}$ avoiding W. W.l.o.g. suppose x_1 is adjacent to w and x_n is adjacent to node $p \in V(\Gamma) \setminus \{w\}$. If $n = 1$, x_1 is either a twin of a node in Γ or is a strongly adjacent node of Type a, b, c, or d[2.1] relative to Γ. Henceforth we assume that $n \geq 2$. This implies that x_1 is not a twin of a node in Γ and x_1 is not a strongly adjacent node of Type a, b, c or d[2.1] relative to Γ.

Lemma 4.1 *In $G \setminus F$, every direct connection $X = x_1, \ldots, x_n$, $n \geq 2$, between $w \in V(P)$ and $V(\Gamma) \setminus \{w\}$ avoiding W is of one of the following types, see Figure 3.*

Type 1 *Node x_n has all its neighbors in $V(P)$. If x_n is not strongly adjacent to Γ, let $p \in V(P)$ be its neighbor. Then either p is adjacent to w or p and w belong to the same side of the bipartition. If x_n is strongly adjacent to Γ, then either x_n is adjacent to w or x_n and w belong to the same side of the bipartition.*

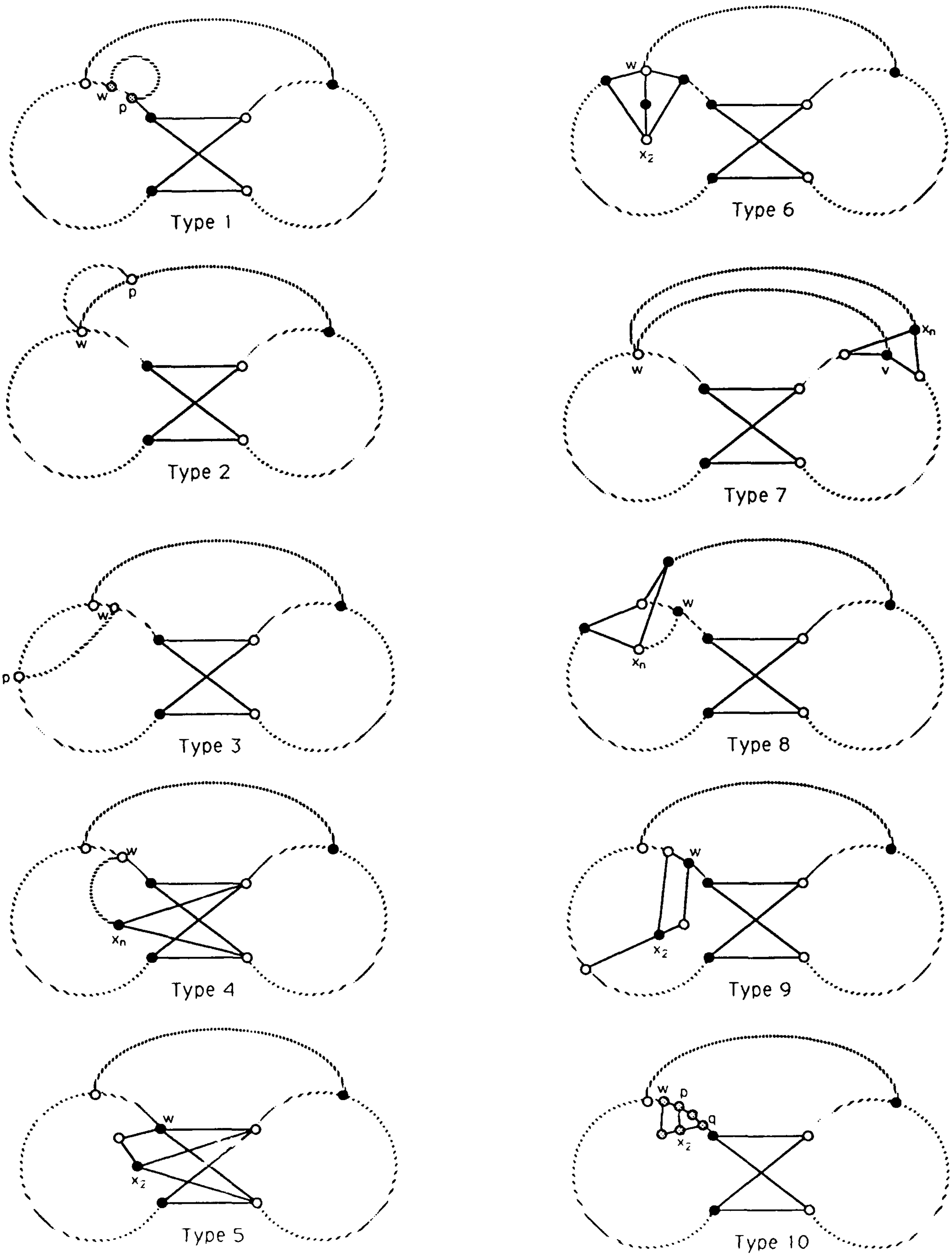


Figure 3: Direct Connections from $w \in V(P)$

Type 2 *Node x_n has all its neighbors in $V(T)$ and $w = h$. If x_n is not strongly adjacent to Γ, let $p \in V(T)$ be its neighbor. Then either $p = k$ or $p \in V^r$. If x_n is strongly adjacent to Γ, then either x_n is adjacent to w or $x_n \in V^r$.*

Type 3 *Node x_n has all its neighbors in $V(S)$. If x_n is not strongly adjacent to Γ, let $p \in V(S)$ be its neighbor. Then either $p = j$ and $w = h$, or $p \in V^r$. If x_n is strongly adjacent to Γ, then either x_n is adjacent to h and $w = h$, or $x_n \in V^r$.*

Type 4 *Node $w \in V^r$ and x_n is a strongly adjacent node of Type a[2.1], adjacent to nodes a and b.*

Type 5 *Node $w = x$, $n = 2$ and x_2 is a strongly adjacent node of Type a[2.1], adjacent to nodes a and b.*

Type 6 *Node $w = h$, $n = 2$ and x_2 is a strongly adjacent node of Type b or c[2.1], adjacent to exactly two nodes in the set $\{i, j, k\}$.*

Type 7 *Node $w = h$ and x_n is a strongly adjacent node of Type b[2.1], adjacent to the neighbor of v in Q and R.*

Type 8 *Node $w \in V^c$ and x_n is a strongly adjacent node of Type c[2.1] adjacent to nodes j and k.*

Type 9 *Node $w \in V^c$, $n = 2$ and x_2 is a strongly adjacent node of Type d[2.1] adjacent to a node in $V(S)$ and one of the neighbors of w in $V(P)$.*

Type 10 *$n = 2$, node x_2 has only two neighbors p and q in Γ, both belonging to the same path P, S or T. Node p is adjacent to w and has degree 2 in Γ.*

Proof: If x_n is a twin of a node d in Γ, then we consider the goggles Γ^* obtained by substituting x_n for d and $X^* = x_1, \ldots, x_{n-1}$ for X. If $n = 2$, i.e. the path X^* contains a unique node, then by Remark 3.11 node x_2 is a twin of a node of degree 2 in Γ, since x_1 is not adjacent to d. Let p and q be the neighbors of x_2 in Γ. Nodes p and q do not belong to $V(Q) \cup V(R)$ otherwise x_2 violates Theorem 2.1 in Γ^*. By Remark 3.11, node x_1 is not of Type a, b, c or d[2.1] in Γ^* nor a twin of h or x in Γ. Hence x_1 is a twin of a node of degree 2 in Γ^*. This yields a path of Type 10. If $n \geq 3$ then in Γ^*, node x_{n-1} is not a strongly adjacent node.

Therefore we assume w.l.o.g. that x_n is not a twin of a node in Γ. There are two cases to consider, depending upon whether x_n is strongly adjacent to Γ or not.

Case 1 Node x_n is not strongly adjacent to Γ.

If $p \in V(P)$ we have a Type 1 path.

Suppose $p \in V(T)\backslash\{h, v\}$. If $w \neq i, h$, replacing P by $P^* = x, P_{xw}, w, X, p$, S by $S^* = u, S, h, T_{hp}, p$ and T by $T^* = p, T_{pv}, v$ we have goggles with a shorter top. If $w = i$, we must have $p = k$, for otherwise we have a $3PC(a, i)$. Now since $x_1 \neq x_n$, we have a parachute with long top and long sides. If $w = h$ we have a Type 2 path.

Suppose $p \in V(Q) \cup V(R)$. Assume w.l.o.g. that $p \in V(Q)$. If $p = a$ and $w \neq x$ there is a $3PC(a, v)$. Since $p = a$ and $w = x$ is impossible by Lemma 1.2, it follows that $p \neq a$. If $w \neq h, i$, we have a $3PC(b, v)$. If $w = i$, then $p = v$ and $h \in N(v)$, for otherwise we have a $3PC(a, i)$. But $w = i$, $p = v$ and $h \in N(v)$ implies that G contains a parachute with long top and long sides. If $w = h$, then h is in $N(v)$, for otherwise we have a $3PC(h, v)$. Furthermore $p = v$ or $p \in N(v)$, for otherwise we have a $3PC(x, h)$. If $p \in N(v)$, G contains a parachute with long top and long sides. Hence $w = h$ implies that $h \in N(v)$, $p = v$ and we have a Type 2 path.

Suppose $p \in V(S)$. If $p = u$, then $w = x$, for otherwise there is a $3PC(u, h)$. But since $x_1 \neq x_n$, $p = u$ and $w = x$ implies that G contains a parachute with long top and long sides. Hence $p \neq u$. If $w \in V^c$, then $w = i$, for otherwise we have a $3PC(w, h)$. Now $w = i$ implies that $p = j$, for otherwise we have a $3PC(a, i)$. But since $x_1 \neq x_n$, $w = i$ and $p = j$ implies that G contains a parachute with long top and long sides. Hence $w \in V^r$. If $p \in V^r$, then we have a Type 3 path. Now if $p \in V^c$, we have a $3PC(w, p)$, unless $w = h$ and $p = j$ in which case we have a Type 3 path.

Case 2 Node x_n is strongly adjacent to Γ.

Suppose x_n is a twin of a node in Γ, say d. Then, modify Γ by replacing d by x_n and consider the direct connection $X^* = x_1, x_2, \ldots, x_{n-1}$. Now it is possible that $x_1 = x_{n-1}$, in which case, with respect to the modified goggles Γ^*, x_1 is either a twin of a node in Γ^* or x_1 is a strongly adjacent node of Type a, b, c, or d[2.1]. If $x_1 \neq x_{n-1}$, with respect to Γ^*, we have the various paths as in Case 1.

Suppose x_n is not a twin of a node in Γ. We now have four subcases.

Case 2.1 Node x_n is a Type a[2.1] strongly adjacent node.

If x_n is adjacent to x and u, by Lemma 1.2, $w \neq x$ and we have a $3PC(u,h)$. Suppose x_n is adjacent to a and b. If $w \in V^r$, we have a Type 4. If $w \in V^c \setminus \{x\}$, we have a $3PC(a,w)$. If $w = x$, by Lemma 1.2, x_1 is adjacent to x_n and we have a Type 5 path.

Case 2.2 Node x_n is a Type b[2.1] strongly adjacent node.

Suppose x_n is adjacent to i and j. By Lemma 1.2, $w \neq i$. If $w \neq h$, we have a $3PC(b,j)$. Hence $w = h$ and, by Lemma 1.2, x_1 is adjacent to x_n and we have a Type 6 path. Suppose x_n is adjacent to the two neighbors of v, one in Q and one in R. If $w \in V^c$, there is a $3PC(a,w)$. Hence $w \in V^r$. If $w \neq h$, we have connected squares. So $w = h$ and we have a Type 7 path.

Case 2.3 Node x_n is a Type c[2.1] strongly adjacent node.

Suppose x_n is adjacent to k and i. By Lemma 1.2, $w \neq i$. If $w \neq h$, we have a $3PC(b,k)$. Hence $w = h$ and, by Lemma 1.2, x_1 is adjacent to x_n and we must have a path of Type 6.

Suppose x_n is adjacent to k and j. If $w = h$, by Lemma 1.2, x_1 is adjacent to x_n and we must have a path of Type 6. If $w \in V^r \setminus \{h\}$, we have a $3PC(w,k)$. Hence if $w \neq h$, then $w \in V^c$ and we have a path of Type 8.

Suppose x_n is adjacent to $p \in V(T) \cap N(v)$ and to $t \in (V(Q) \cup V(R)) \cap N(v)$. Assume w.l.o.g. that $t \in V(Q) \cap N(v)$. If $h \notin N(v)$, that is $h \neq p$, or if $h \neq w$, we have a $3PC(x,t)$. So $h = p = w$ and the path $h, x_1, x_2, \ldots, x_n$ violates Lemma 1.2.

Case 2.4 Node x_n is a Type d[2.1] strongly adjacent node.

Now $V(Q) \cap N(x_n) = \emptyset$, for otherwise we have a $3PC(x_n, v)$. Hence $V(P) \cap N(x_n) \neq \emptyset$. Let p and t be the unique neighbors of x_n in P and S respectively. If $p = w$, we have $x_1 = x_n$ contradicting the assumption $n \geq 2$. If $p \neq w$, then $p \in N(w)$, for otherwise we have a $3PC(x_n, h)$. Now x_1 must be adjacent to x_n and we have a path of Type 9, for otherwise we have a parachute with long top and long sides. □

5 Direct Connections from a Strongly Adjacent Node of Type d

Let $w \in V^c$ be a Type d[2.1] node adjacent to $p \in V(P)$ and $s \in V(S)$. Let W be the set of Type d[2.1] nodes, distinct from w, which are adjacent to a node in P and a node in S. In the partial graph $G \setminus \{wp, ws\}$, let $X = x_1, \ldots, x_n$ be a direct connection between w and $V(\Gamma)$ avoiding W. W.l.o.g. suppose x_1 is adjacent to w and x_n is adjacent to node $t \in V(\Gamma)$.

Lemma 5.1 *In $G \setminus \{wp, ws\}$, every direct connection X between w and $V(\Gamma)$ avoiding W is one of the following types, see Figure 4.*

Type 1 *Node x_1 is a twin of p or s.*

Type 2 *Node x_1 is not strongly adjacent to Γ but x_1 is adjacent to t which is a neighbor of p or s in Γ or node x_2 is a twin of a neighbor of p or s in Γ.*

Type 3 *Node x_n is not strongly adjacent to Γ and its neighbor in Γ is $t = p$ or s.*

Proof: Suppose first $n = 1$. Assume x_1 is strongly adjacent to Γ. If x_1 has two or three neighbors in the set $\{i, j, k\}$, there is an odd wheel with center x_1. Now it follows that x_1 must be a twin of p or s for otherwise there is a $3PC(w, h)$. This yields a path of Type 1. If x_1 is not strongly adjacent to Γ, t must be a neighbor of p or s for otherwise there is a $3PC(w, h)$. This yields a path of Type 2.

Suppose now $n \geq 2$. There are two cases to consider, depending upon whether x_n is strongly adjacent to Γ or not.

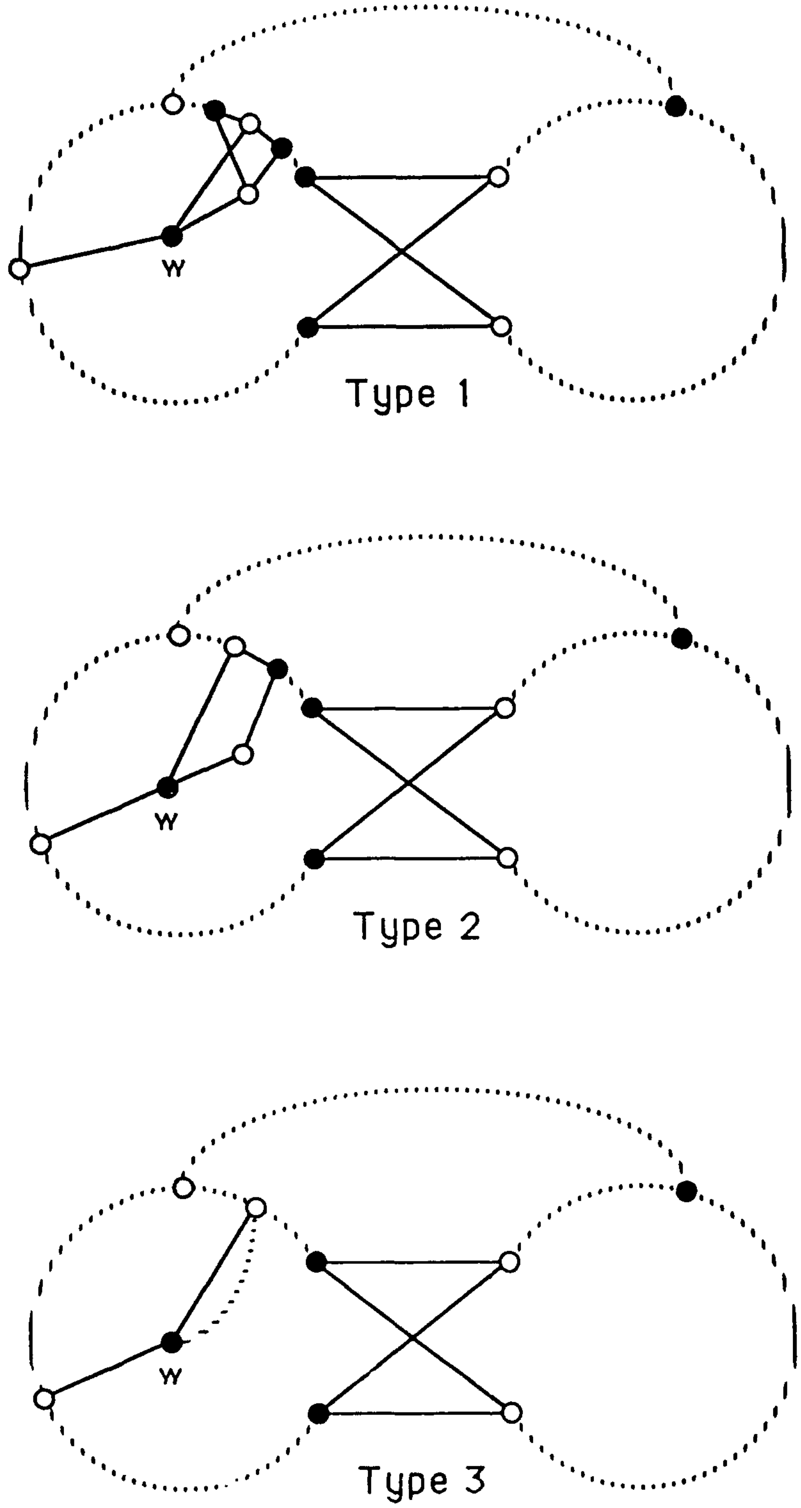


Figure 4: Direct Connections from a Type d Node

Case 1 Node x_n is not strongly adjacent to Γ.

If $t = p$ or s, we have path of Type 3. If t is not adjacent to p or s and $t \neq p, s$, there is a $3PC(w, h)$. If t is adjacent to p or s, we have a parachute with long top and long sides.

Case 2 Node x_n is strongly adjacent to Γ.

Suppose x_n is a twin of a node in Γ, say d. Then, modify Γ by replacing d by x_n and consider the direct connection $X^* = x_1, \ldots, x_{n-1}$. If $x_1 = x_{n-1}$ then, with respect to the modified goggles, the path X^* must be of Type 2. By Lemma 3.10, w is adjacent to all the twins of p and s. Now $x_1 = x_{n-1}$, otherwise there is a parachute with long top and long sides or a $3PC(w, h)$.

Suppose x_n is not a twin of a node in Γ. There are four subcases.

Case 2.1 Node x_n is a Type a[2.1] strongly adjacent node.

There is a $3PC(w, h)$.

Case 2.2 Node x_n is a Type b[2.1] strongly adjacent node.

If x_n is adjacent to i and j, there is a $3PC(w, x_n)$. If x_n is adjacent to the neighbors of v in Q and R, there is a $3PC(w, a)$.

Case 2.3 Node x_n is a Type c[2.1] strongly adjacent node.

If x_n is adjacent to k and either i or j, there is a $3PC(w, x_n)$. If x_n is adjacent to the neighbor of v in T and to a neighbor of v in Q or R, there is a $3PC(w, a)$.

Case 2.4 Node x_n is a Type d[2.1] strongly adjacent node.

There is a $3PC(w, h)$. □

6 Direct Connections from a Strongly Adjacent Node of Type a

Let $w \in V^c$ be a Type a[2.1] node adjacent to a and b. Let $W(w)$ be the set consisting of twins of a and b and Type a[2.1] nodes, but not nodes a, b and w. In the partial graph $G \setminus \{wa, wb\}$, let $X = x_1, \ldots, x_n$ be a direct connection

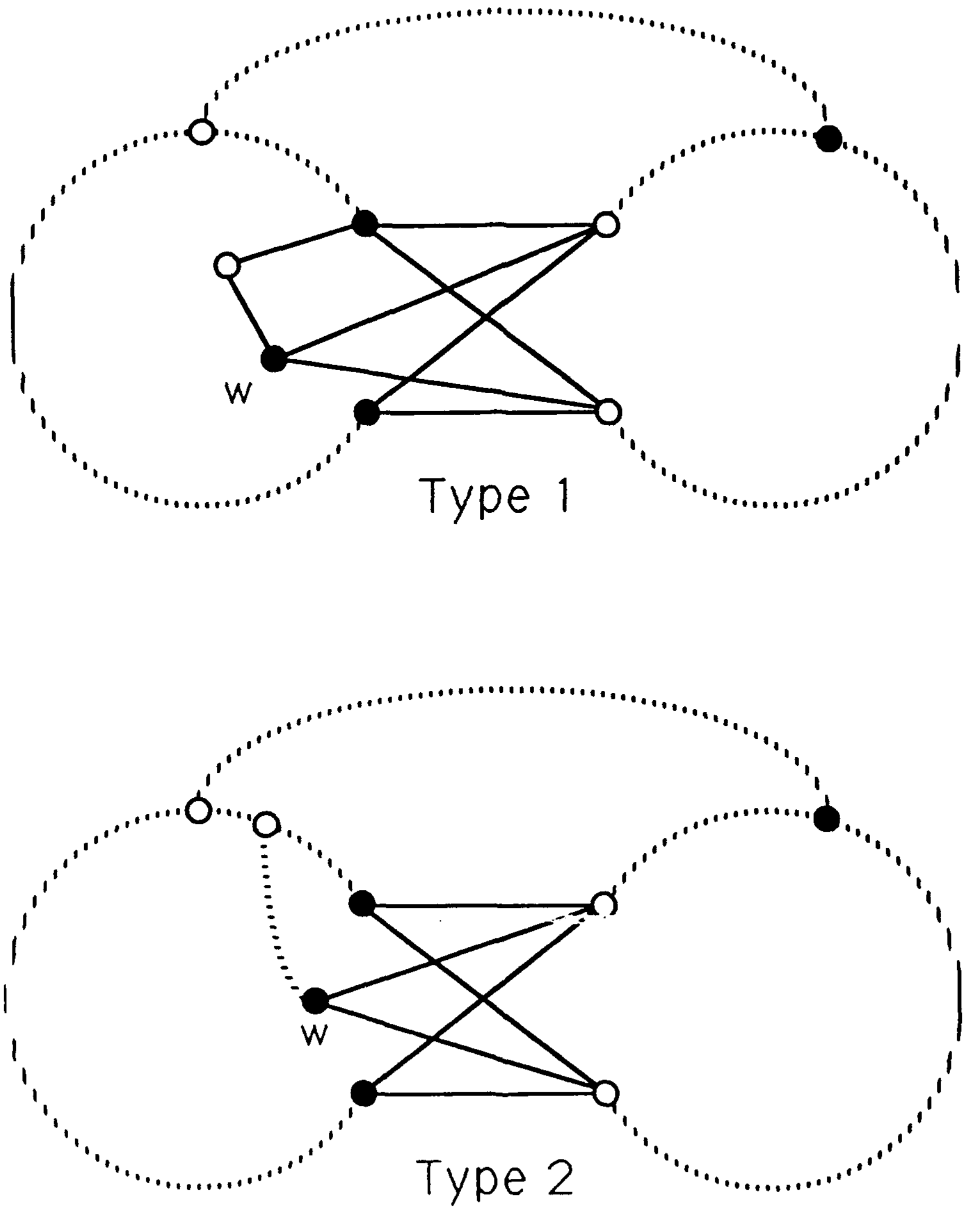


Figure 5: Direct Connections
from a Type a Node

between w and $V(\Gamma)$ avoiding $W(w)$. W.l.o.g. suppose x_1 is adjacent to w and x_n is adjacent to $t \in V(\Gamma)$.

Lemma 6.1 *In $G \setminus \{wa, wb\}$, every direct connection X between w and $V(\Gamma)$ avoiding $W(w)$ is one of the following types, see Figure 5.*

Type 1 *Either $n = 1$ and node x_1 is adjacent to u or x but not strongly adjacent to Γ. Or $n = 2$ and node x_2 is a twin of u or x.*

Type 2 *Node x_n is not strongly adjacent to Γ and $t \in V^r \cap (V(P) \cup V(S))$ or node $x_n \in V^r$ is a twin of a node in $V(P) \cup V(S)$.*

Proof:

Case 1 Node x_n is not strongly adjacent to Γ.

If $t = u$ or x then $n = 1$, for otherwise there is a violation of Lemma 1.2. This yields a path of Type 1. Suppose $t \neq u$, x. If $t = a$ or b there is a violation of Lemma 1.2. If $t \in V(Q) \cup V(R) \setminus \{a, b\}$, there is a $3PC(a, v)$ or a $3PC(b, v)$. If $t \in V(T) \setminus \{h, v\}$, there are goggles with a shorter top. If $t \in V(P) \cup V(S)$, it follows that $t \in V^r$ for otherwise there a $3PC(a, t)$ or $3PC(b, t)$. This yields a path of Type 2.

Case 2 Node x_n is strongly adjacent to Γ.

Case 2.1 $n = 1$.

Suppose x_1 is a twin of a node in Γ. If x_1 is a twin of h, then w is a strongly adjacent node of Type f[2.1] in the goggles obtained from Γ by replacing h with x_1, contradicting Theorem 2.2. So x_1 has two neighbors in Γ. If x_1 is adjacent to Q or R, there is a $3PC(a, v)$ or a $3PC(b, v)$. If x_1 is adjacent to T, there are goggles with a shorter top. If x_1 is adjacent to P or S, there are goggles with a fewer nodes, unless x_1 is adjacent to x or u. Suppose x_1 is adjacent to x or u. W.l.o.g. assume that x_1 is adjacent to x. Since x_1 is a twin of the neighbor of x, say d, in P, from Lemma 3.2 it follows that w is adjacent to d. Then w is a twin of x and not a Type a[2.1] node.

Suppose x_1 is a Type b[2.1] node. Since $x_1 \in V^r$, x_1 is adjacent to the neighbors i and j and there is a $3PC(a, i)$.

Suppose x_1 is a Type c[2.1] node. W.l.o.g. assume x_1 is adjacent to i and k. Then there is a $3PC(a,i)$.

Suppose x_1 is a Type d[2.1] node. We have a violation of Lemma 5.1.

Case 2.2 $n \geq 2$.

Suppose x_n is a twin of a node in Γ, say d. Then, we modify Γ by replacing d by x_n and consider the direct connection $X^* = x_1, \ldots, x_{n-1}$. Now we are back to Case 1 with respect to the modified goggles.

Suppose x_n is not a twin of a node in Γ. We have three subcases.

Case 2.2.1 Node x_n is a Type b[2.1] strongly adjacent node.

If x_n is adjacent to i and j, there is a $3PC(a,i)$. If x_n is adjacent to the neighbors of v in Q and R, say t and t', there are connected squares with $P_1 = x, P, h, T, v$; $P_2 = w, x_1, X, x_n$; $P_3 = a, Q_{at}, t$; $P_4 = b, R_{bt'}, t'$.

Case 2.2.2 Node x_n is a Type c[2.1] strongly adjacent node.

If x_n is adjacent to k and i (j resp.), there is a $3PC(a,i)$ ($3PC(a,j)$ resp.). If x_n is adjacent to the neighbor of v in T and to a neighbor of v in Q or R, there is a $3PC(a,x_n)$ or a $3PC(b,x_n)$.

Case 2.2.3 Node x_n is a Type d[2.1] strongly adjacent node.

Lemma 5.1 is violated. □

Lemma 6.2 *In $G \setminus \{wa, wb\}$, there exists a direct connection of Type 2[6.1].*

Proof: Suppose there exists a direct connection of Type 1[6.1] with node x_1 adjacent to w and x or a twin of x. Then there does not exist a direct connection of Type 1[6.1] with node y_1 adjacent to w and u or a twin of u, otherwise there is an odd wheel with center a.

Assume w.l.o.g. that Type 1[6.1] paths, if they exist, have their unique node adjacent to x or a twin of x. Consider the parachute with center node x, middle path x, P, h, T, v, side paths Q and R and top path a, w, b. By Corollary 1.1, there is a direct connection of Type d[3.3(III)] or d1[4.1(III)] from the middle path of this parachute to w. Let $Y = y_1, \ldots, y_n$ be this direct connection, where y_1 is adjacent to w. If some node in $\{y_1, \ldots, y_{n-1}\}$ is adjacent to a node in S, let y_m be such a node with the lowest index. Now, the path $y_1, \ldots, y_m$ must satisfy Lemma 6.1 and therefore it is of Type 2[6.1], since no path of Type 1[6.1] is adjacent to a node of S by our assumption. Hence, no node in $\{y_1, \ldots, y_{n-1}\}$ is adjacent to a node in S. Therefore, it follows from Lemma 6.1 that Y is of Type 2[6.1]. □

Lemma 6.3 *Suppose the top path T of Γ is of length greater than 1. Let w be a Type a[2.1] node adjacent to a and b, and let y be a Type a[2.1] node adjacent to u and x. Then w and y are adjacent.*

Proof: Suppose w and y are not adjacent. By Lemma 6.2, there exist a direct connection $X = x_1, \ldots, x_n$ of Type 2[6.1] from w to P or S, say P, and a direct connection $Y = y_1, \ldots, y_m$ of Type 2[6.1] from y to Q or R, say Q. The only possible adjacency between a node in X and a node in Y is between x_n and y_m for otherwise there is a violation of Lemma 6.1. If x_n and y_m are adjacent, there is a $3PC(h, v)$. If x_n and y_m are not adjacent, there is a $3PC(u, h)$. Hence w and y are adjacent. □

Lemma 6.4 *Suppose the top path T of Γ is of length 1. If w is a Type a[2.1] node adjacent to a and b, and y is a Type a[2.1] node adjacent to u and x but not to w, then every direct connection $X = x_1, \ldots, x_n$ between w and $V(\Gamma)$ avoiding $W(w)$ is of Type 2[6.1] and x_n is adjacent to h. Similarly, every direct connection $Y = y_1, \ldots, y_m$ between y and $V(\Gamma)$ avoiding $W(y)$ is of Type 2[6.1] and y_m is adjacent to v. Furthermore x_n and y_m are adjacent and this is the only adjacency between a node of X and a node of Y.*

Proof: By Lemma 6.2, there exists a direct connection $Y = y_1, \ldots, y_m$ of Type 2[6.1] from y and a direct connection $X = x_1, \ldots, x_n$ of Type 2[6.1] from w. Suppose x_n is adjacent to $t \neq h$ in P or S, say in P. Since w and y are not adjacent, the only possible adjacency between a node in X and a node in Y is between x_n and y_m for otherwise there is a violation of Lemma 6.1. If x_n and y_m are adjacent, there is a $3PC(a, x_n)$. If x_n and y_m are not adjacent, there is a $3PC(u, h)$. So x_n must be adjacent to h. By symmetry, y_m must be adjacent to v. It also follows that x_n and y_m are adjacent, otherwise there is a $3PC(u, h)$.

Now suppose there exists a Type 1[6.1] direct connection with unique node z_1 adjacent to w and x. Node z_1 is not adjacent to any of the nodes in Y, otherwise there is a violation of Lemma 6.1. Now we have a violation of Lemma 1.2 for the following parachute. The top path is a, x, y, the side paths are Q and y, Y, v, the center node is u and the middle path is u, S, h, v. The extra path is x, z_1, w, a. □

Lemma 6.5 *Suppose w is a Type a[2.1] node adjacent to a and b, and y is a Type a[2.1] node adjacent to u and x. Suppose w and y are adjacent. Let $N(\Gamma)$ be the set of nodes adjacent to at least one node of Γ. In the partial graph $G \setminus \{wy\}$, there cannot be a direct connection between w and y avoiding $V(\Gamma) \cup N(\Gamma) \setminus \{w, y\}$.*

Proof: By Lemma 6.2, there exist a direct connection $X = x_1, \dots, x_n$ of Type 2[6.1] from w to P or S, say P, and a direct connection $Y = y_1, \dots, y_m$ of Type 2[6.1] from y to Q or R, say Q. The only possible adjacency between a node in X and a node in Y is between x_n and y_m for otherwise there is a violation of Lemma 6.1. If x_n and y_m are adjacent, there is a $3PC(y, x_n)$. Hence no node of Y is adjacent to a node in X. Assume that x_n is adjacent to $p \in P$ and y_m is adjacent to $q \in Q$. In the partial graph $G \setminus \{wy\}$ suppose there exists a direct connection $L = l_1, \dots l_t$ between w and y avoiding $V(\Gamma) \cup N(\Gamma) \setminus \{w, y\}$. Now no node of L is adjacent to a node of Y or X, otherwise there is a violation of Lemma 6.1. Now we have a violation of Lemma 1.2 for the following parachute. The top path is b, w, y, the side paths are R and $y, y_1, Y, y_m, q, Q_{q,v}, v$, the center node is u and the middle path is u, S, h, T, v. The extra path is L. □

Lemma 6.6 *Suppose w and w^* are two Type a[2.1] nodes adjacent to a and b. Suppose y is a Type a[2.1] node adjacent to u and x. Suppose w^* and y are adjacent but w and y are not adjacent. Let $N(\Gamma)$ be the set of nodes adjacent to at least one node of Γ. Then there cannot be a direct connection between w and w^* avoiding $V(\Gamma) \cup N(\Gamma)$.*

Proof: By Lemma 6.3, the top path T of Γ must be of length 1. By Lemma 6.4, there exist a direct connection $X = x_1, \dots, x_n$ of Type 2[6.1] from w to h and a direct connection $Y = y_1, \dots, y_m$ of Type 2[6.1] from y to v. Assume w.l.o.g that x_n is adjacent to h and y_m is adjacent to v. By Lemma 6.4, x_n and y_m are adjacent and this is the only adjacency between a node in X and a node in Y. No node of Y is adjacent to w^*, otherwise there is a violation of Lemma 6.1. No node of X is adjacent to w^*, otherwise there is a $3PC(y, x_n)$. Now suppose there exists a direct connection L, between w and w^* avoiding $V(\Gamma) \cup N(\Gamma)$. Now we have a violation of Lemma 1.2 for the following parachute. The top path is b, w^*, y, the side paths are R and y, y_1, Y, y_m, v, the center node is u and the middle path is u, S, h, v. The extra path is b, w, L, w^*. □

7 Partition of the Neighbors of h

Assume w.l.o.g. that, if a Type b[2.1] node exists, there is one adjacent to i and j. If no Type b[2.1] node exists but a Type c[2.1] node exists, assume w.l.o.g. that there is a Type c[2.1] node adjacent to i and k.

Let $Z(h)$ comprise

- the set of nodes h, i, j, k, their twins relative to Γ,
- the nodes of $N(h)$
- the Type b[2.1] nodes adjacent to i and j,
- the Type c[2.1] nodes adjacent to i and k.

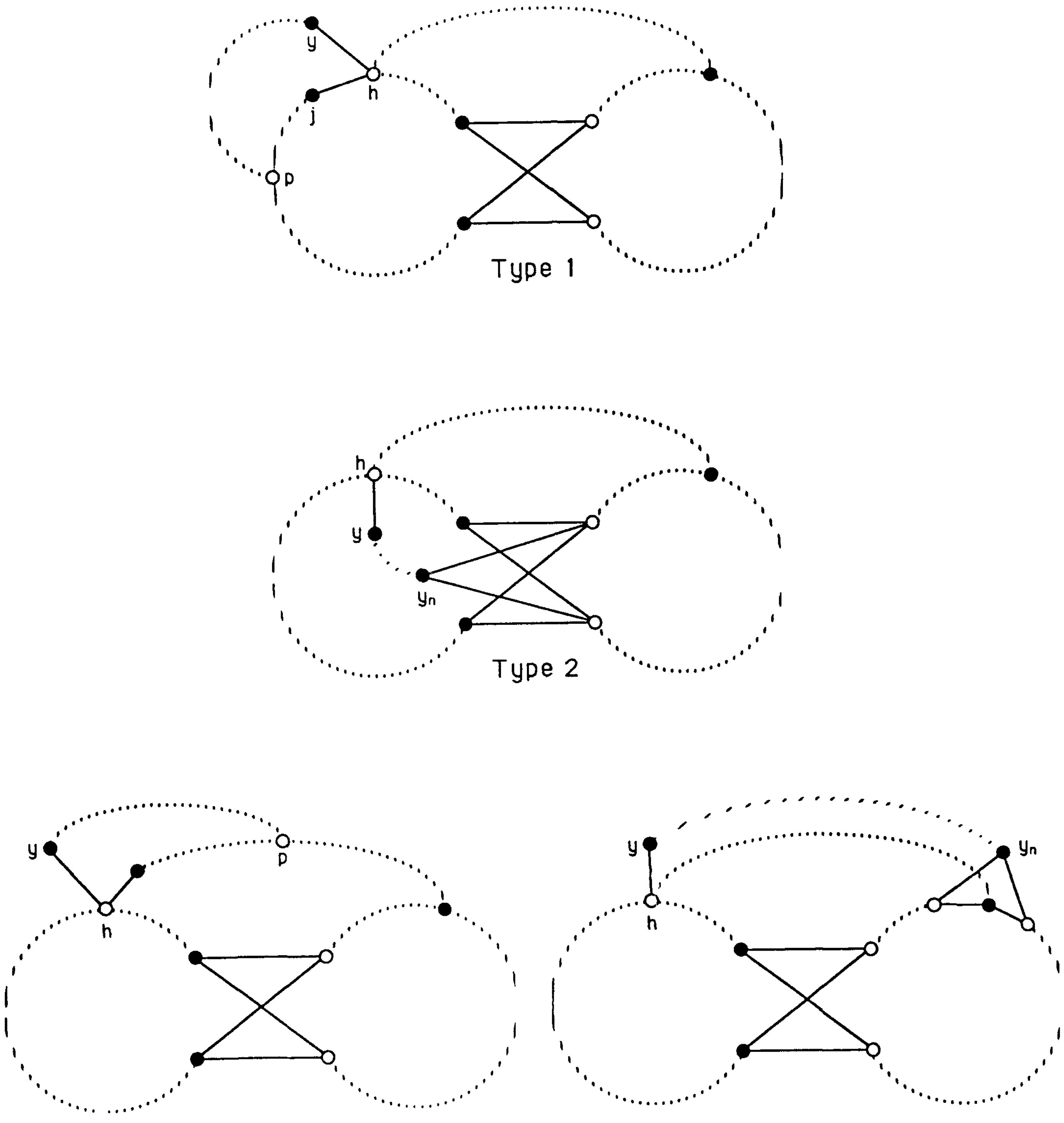
By Lemma 3.5 and Remarks 3.9 and 3.13, it follows that $Z(h)$ is an extended star.

Let $y \in N(h) \setminus i, j, k$ be a node which not strongly adjacent to Γ.

There must be a direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$. Assume that y_1 is adjacent to y and y_n is adjacent to $p \in V(\Gamma) \setminus \{h, i, j, k\}$. Note that the nodes $y_1, \ldots, y_n$ can be adjacent to i, j or k but none is adjacent to two nodes in the set $\{i, j, k\}$.

Lemma 7.1 *Suppose the top path T of Γ is of length greater than 1. Every direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$ is one of the following types. see Figure 6.*

Type 1 *Node y_n is adjacent to $p \in V(P) \cup V(S) \setminus \{h, i, j\}$ but is not of Type a or d[2.1]. No node of Y is adjacent to k. If $p \in V(P)$ ($V(S)$ resp.), then no node of Y is adjacent to j (i resp.).*



Type 3

Figure 6: Direct Connections from a Neighbor of h. (Long Top)

Type 2 *Node y_n is adjacent to a and b and no node of Y is adjacent to i, j or k.*

Type 3 *Node y_n is adjacent to $p \in V(T) \setminus \{h, k\}$ but is not of Type c[2.1], or y_n is a Type b[2.1] node adjacent to the neighbors of v in Q and R. If $p \in V(T) \setminus \{h, k\}$, then no node of Y is adjacent to i or j. If y_n is a Type b[2.1] node, then no node of Y is adjacent to i, j, k.*

Proof:

Case 1 Node y_n is not strongly adjacent.

Suppose $p \in V(T) \setminus \{h, k\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. Hence this yields a path of Type 3.

Suppose $p \in V(Q) \cup V(R) \setminus \{v\}$. W.l.o.g. assume $p \in V(Q) \setminus \{v\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. If none of the nodes in Y is adjacent to k, there is a $3PC(h, v)$. So, let t be the largest index such that y_t is adjacent to k. If $p \in V^r$, there is a $3PC(k, p)$. So p is not adjacent to v and there are goggles with a shorter top path h, k obtained from Γ by replacing Q by $a, Q_{ap}, p, y_n, Y_{y_n y_t}, y_t, k$ and R by b, R, v, T_{vk}, k.

Suppose $p \in V(P) \cup V(S) \setminus \{h, i, j\}$. W.l.o.g. assume $p \in V(P) \setminus \{h, i\}$. If any of the nodes in Y is adjacent to i or k, there is a $3PC(a, j)$ or $3PC(a, k)$. This yields a path of Type 1.

Case 2 Node y_n is strongly adjacent.

Case 2.1 Node y_n is a twin of a node of Γ.

Suppose y_n is a twin of $d \in V(\Gamma)$. If $n = 1$, it follows that y_1 must be adjacent to i, j or k, for otherwise replacing d by y_1 yields a violation of Theorem 2.2. If y_1 is adjacent to k, we get a path of Type 3, otherwise we get a path of Type 1. If $n \geq 2$, replacing d by y_n, we are back in Case 1 with respect to the modified goggles.

Case 2.2 Node y_n is a Type a[2.1] node.

Suppose y_n is adjacent to x and u. There is a contradiction to Lemma 6.1, irrespective of whether any of the nodes in Y is adjacent to i, j or k.

Suppose y_n is adjacent to a and b. If any of the nodes in Y is adjacent to i, j or k, we have a violation of Lemma 6.1. Otherwise we have a Type 2 path.

Case 2.3 Node y_n is a Type b[2.1] node adjacent to the neighbors of v in Q and R.

No node in Y is adjacent to i or j, otherwise there is a $3PC(a,i)$ or $3PC(a,j)$. Now no node in Y is adjacent to k, otherwise there is a $3PC(p,k)$. This yields a path of Type 3.

Case 2.4 Node y_n is a Type c[2.1] node adjacent to the neighbors of v in Q and T.

No node in Y is adjacent to i or j, otherwise there is a $3PC(a,i)$ or $3PC(a,j)$. Assume $p \in V(Q)$. Now there is a $3PC(p,x)$, irrespective of whether a node of Y is adjacent to k or not.

Case 2.5 Node y_n is a Type d[2.1] node.

There is a violation of Lemma 5.1. □

Lemma 7.2 *Suppose the top path T of Γ is of length greater than 1.*

- *If there exists a Type 1[7.1] direct connection between y and $V(\Gamma) \setminus \{h,i,j,k\}$ avoiding $Z(h) \setminus \{y\}$, there exists one, say $X = x_1, \ldots, x_m$ such that no node in the set $\{x_1, \ldots, x_{m-1}\}$ is adjacent to i, j or k.*
- *If there exists a Type 3[7.1] direct connection between y and $V(\Gamma) \setminus \{h,i,j,k\}$ avoiding $Z(h) \setminus \{y\}$, there exists one, say $X = x_1, \ldots, x_m$ such that no node in the set $\{x_1, \ldots, x_{m-1}\}$ is adjacent to i, j or k.*

Proof: To prove the first part of the lemma, let $Y = y_1, \ldots, y_n$ be a direct connection of Type 1[7.1]. Assume w.l.o.g. that y_n is adjacent to $p \in V(P)$. If y_n is a twin of a node d in Γ, then replace d by y_n. So we assume w.l.o.g. that y_n is not strongly adjacent to Γ. If the nodes of Y are not adjacent to node i, the result follows from Lemma 7.1. If node i has two or more neighbors in Y, there is a wheel with center i. So node i has exactly one neighbor in Y, say y_l, and node p is not adjacent to i. If $l > 1$, there is a parachute with long top and long sides: the middle node is i, the side nodes are h and y_l, the bottom node is p. So $l = 1$ and Corollary 1.1 can be applied to the parachutes with top path h, y, y_1 and middle path P_{ip}, the subpath of P connecting i to p. There exists a direct connection X of Type d[3.3(III)] or d1[4.1(III)], connecting $V(P_{ip})$ to node y. The various possible choices for the side path connecting h to p imply that $V(X)$ and $V(\Gamma) \setminus V(P_{ip})$ do not have common or adjacent nodes. This proves the result. The second part of the lemma is proved by an analogous argument. □

Lemma 7.3 *Suppose the top path T of Γ is of length 1. Every direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$ is one of the following types, see Figure 7.*

Type 1 *Node y_n is adjacent to $p \in V(P) \cup V(S) \setminus \{h, i, j\}$ but is not of Type a or d[2.1]. No node of Y is adjacent to v. If $p \in V(P)$ ($V(S)$ resp.), then no node of Y is adjacent to j (i resp.).*

Type 2 *Node y_n is adjacent to a and b and no node of Y is adjacent to i, j or v.*

Type 3 *Node y_n is adjacent to x and u and no node of Y is adjacent to i, j. Node y_1 is the unique node of Y adjacent to v.*

Proof:

Case 1 Node y_n is not strongly adjacent.

Suppose $p \in V(Q) \cup V(R) \setminus \{v\}$. W.l.o.g. assume $p \in V(Q) \setminus \{v\}$. If Y has a node adjacent to i or j, there is a $3PC(a, i)$ or a $3PC(a, j)$. One of the nodes in Y is adjacent to v, otherwise there is a violation of Lemma 4.1. Now node p is not adjacent to v for otherwise we have a wheel with v as center. Let y_t be the unique node of Y which is adjacent to v. If $t \neq 1$, there is a parachute with long top $h, y, y_1, Y_{y_1 y_t}, y_t$, side paths P and $y_t, Y_{y_t y_n}, y_n, p, Q_{pa}, a, x$, center node v and middle path v, R, b, x. Now consider the above parachute Π_1 where $t = 1$. Applying Corollary 1.1, there exists a path $X = x_1, \ldots, x_m$ of Type d[3.3(III)] or d1[4.1(III)] relative to Π_1 where x_m is adjacent to a node in $V(R) \setminus \{v\}$. No node of X can be adjacent to a node of S for otherwise there is a violation of Corollary 1.1 applied to the parachute Π_2 obtained from Π_1 by replacing the side path P by S and the bottom node x by u. It follows that the path X violates Lemma 4.1.

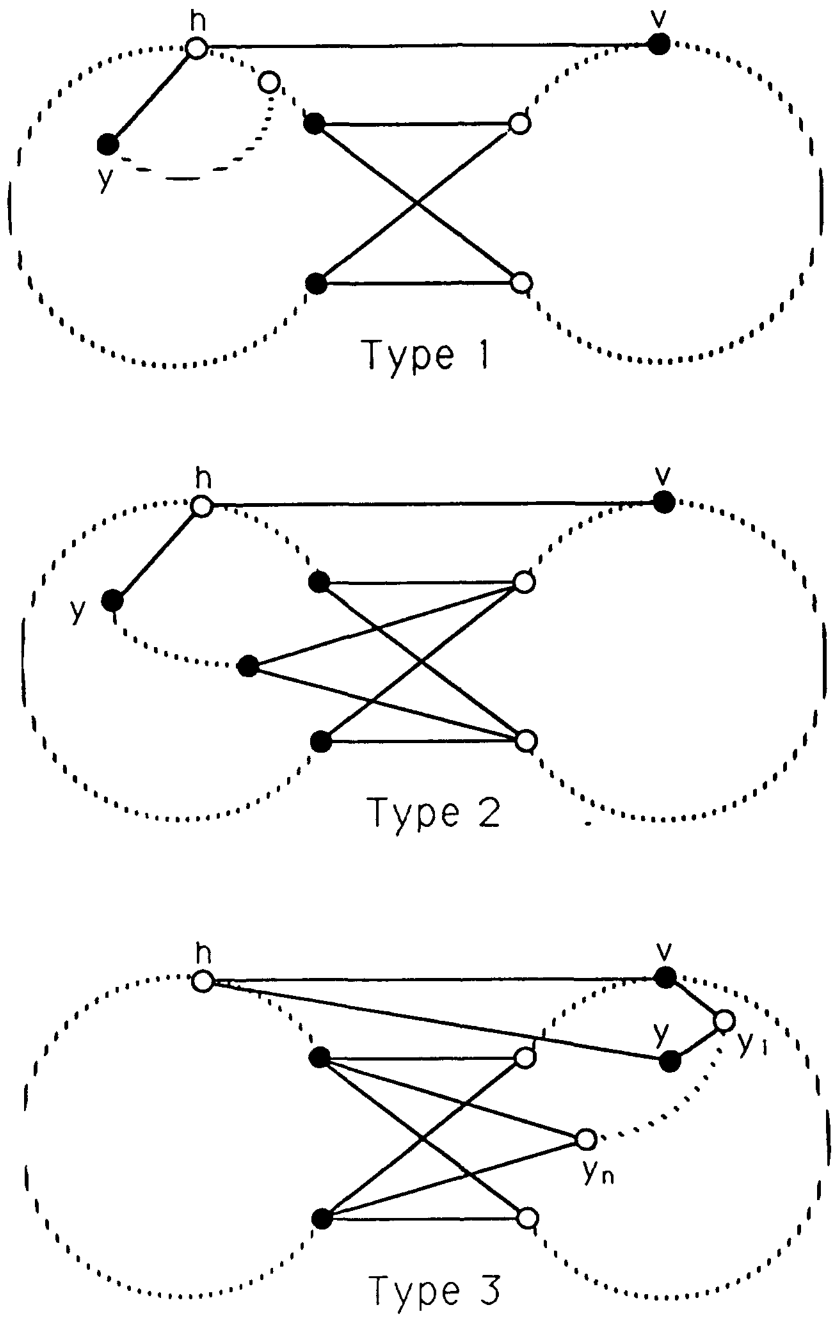


Figure 7: Direct Connections from a Neighbor of h. (Short Top)

Suppose $p \in V(P) \cup V(S) \setminus \{h, i, j\}$. W.l.o.g. assume $p \in V(P) \setminus \{h, i\}$. If any of the nodes in Y is adjacent to j or k, there is a $3PC(a, j)$ or $3PC(a, k)$. This yields a path of Type 1.

Case 2 Node y_n is strongly adjacent.

Case 2.1 Node y_n is a twin of a node of Γ.

Suppose $n = 1$. Let y_n be a twin of $d \in V(\Gamma)$. Node y_1 must be adjacent to i, j or v, for otherwise replacing d by y_1 yields a violation of Theorem 2.2. If y_1 is adjacent to i or j, we have a Type 1 path. Suppose y_1 is adjacent to v. Using parachutes Π_1 and Π_2 as in Case 1 above, we get a violation of Lemma 4.1.

If $n \geq 2$, replacing d by y_n, we are back in Case 1 with respect to the modified goggles.

Case 2.2 Node y_n is a Type a[2.1] node.

Suppose y_n is adjacent to x and u. No node of Y is adjacent to i or j for otherwise there is $3PC(y_n, i)$ or a $3PC(y_n, j)$. Now a node of Y must be adjacent to v otherwise there is a violation of Lemma 6.1. Let y_t be the unique node of Y which is adjacent to v. If $t \neq 1$, there is a parachute with long top $h, y, y_1, Y_{y_1 y_t}, y_t$, side paths P and $y_t, Y_{y_t y_n}, y_n, x$, center node v and middle path v, R, b, x. Hence $t = 1$. This yields a path of Type 3.

Suppose y_n is adjacent to a and b. If any of the nodes in Y is adjacent to i, j or v, we have a violation of Lemma 6.1. Otherwise we have a Type 2 path.

Case 2.3 Node y_n is a Type b[2.1] node adjacent to the neighbors of v in Q and R.

By Lemma 3.5, no such node y_n exists since $|T| = 1$.

Case 2.4 Node y_n is a Type c[2.1] node adjacent to the neighbors of v in Q and T.

Type c[2.1] nodes belong to $Z(h)$ and therefore this case cnnot occur.

Case 2.5 Node y_n is a Type d[2.1] node.

There is a violation of Lemma 5.1. □

Lemma 7.4 *Suppose the top path T of Γ is of length 1 and there is a direct connection $Y = y_1, \ldots, y_n$ of Type 3[7.3] from $y \in N(h) \setminus H$, where y_n is adjacent to x and u. Then there exists a direct connection $X = x_1, \ldots, x_m$ of Type 2[7.3] from y, where x_m is adjacent to a and b. Moreover, x_m is not adjacent to y_n and no node of Y is coincident with or adjacent to a node of X.*

Proof: By Lemma 7.3, y_1 is adjacent to v. Now consider the parachute with top path h, y, y_1, side paths P and $y_1, Y_{y_1 y_n}, x$, center node v and middle path v, Q, a, x. By Corollary 1.1, there must be a Type d[3.3(III)] or Type d1[4.1(III)] direct connection $X = x_1, \ldots, x_m$ from node y to a node of $V(Q) \setminus \{v\}$. Therefore, node x_m is not adjacent to y_n. No node of X can be adjacent to S. Furthermore, x_m must be adjacent to a and b and no node of Y is coincident with or adjacent to a node of X, otherwise there is violation of Lemma 4.1. □

Definition 7.5 *If the top path T of Γ has length greater than 1, let*

$N_{PS}(h) = \{y \in N(h) \setminus H$: there is a Type 1 or Type 2[7.1] direct connection from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}\}$,

$N_{QR}(h) = \{y \in N(h) \setminus H$: there is a Type 3[7.1] direct connection from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}\}$.

If the top path T of Γ has length 1, let

$N_{PS}(h) = \{y \in N(h) \setminus H$: there is a Type 1 or Type 2[7.3] but not Type 3[7.3] direct connection from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}\}$.

$N_{QR}(h) = \{y \in N(h) \setminus H$: there is a Type 3[7.3] direct connection from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}\}$,

Lemma 7.6 *If $y \in N_{PS}(h)$, then $y \notin N_{QR}(h)$.*

Proof: There are two cases to consider depending on the length of T.

Case 1 $|T| > 1$.

Suppose the lemma is false, i.e. there exists a y in $N_{PS}(h)$ and $N_{QR}(h)$. Let $X = x_1, \ldots, x_n$ be a Type 1 or 2[7.1] path and $Y = y_1, \ldots, y_m$ be a Type 3[7.1] path. There are two subcases.

Case 1.1 X is a Type 1[7.1] path.

W.l.o.g. assume that x_n is adjacent to $p \in V(S) \setminus \{h, j\}$. By Lemma 7.2, we can assume w.l.o.g. that $x_1, \ldots, x_{m-1}$ are not adjacent to node j. If x_n is a twin of a node d in Γ, replace d by x_n and X by $x_1, \ldots, x_{m-1}$.

If a node of X is coincident with or adjacent to a node of Y , there is a path from p to k or from p to q violating Lemma 4.1. Hence no node of X is coincident with or adjacent to a node of Y. Now there is a $3PC(a, y)$.

Case 1.2 X is a Type 2[7.1] path.

Node x_n is a Type a[2.1] node adjacent to a and b.

If a node of X is coincident with or adjacent to a node of Y, there is a violation of Lemma 6.1. Now there is a $3PC(a, y)$.

Case 2 $|T| = 1$

Suppose the lemma is false. Let X be a Type 1[7.3] path and $Y = y_1, \ldots, y_m$ be a Type 3[7.3] path where y_m is adjacent to x and u. There cannot be a node of X adjacent to a node of Y otherwise there is a violation of Lemma 4.1. Now there is a $3PC(y_m, y)$.

By the definition of $N_{PS}(h)$, if $y \in N_{PS}(h)$, then y cannot have a Type 3[7.3] path. □

Corollary 7.7 *If $y \in N_{PS}(h)$ then there cannot be a direct connection $Y = y_1, \ldots, y_m$, between y and $V(\Gamma) \setminus \{h, i, j\}$ avoiding $Z(h) \setminus \{y, k\}$, such that y_m is adjacent to k. If $y \in N_{QR}(h)$ then there cannot be a direct connection $X = x_1, \ldots, x_n$, between y and $V(\Gamma) \setminus \{h, k\}$ avoiding $Z(h) \setminus \{y, i, j\}$, such that x_n is adjacent to i or j.*

Proof: Suppose the contrary. Then there exists a direct connection $Y = y_1, \ldots, y_m$, between $y \in N_{PS}(h)$ and $V(\Gamma) \setminus \{h, i, j\}$ avoiding $Z(h) \setminus \{y, k\}$, such that y_m is adjacent to k. If y_m is strongly adjacent to Γ, it follows that y_m must be a twin of the neighbor of k, say $t \neq h$ in T. Then $y \in N_{QR}(h)$ and there is a violation of Lemma 7.5. Hence y_m is not strongly adjacent to Γ. Since $Z(h)$ is an extended star but not a cutset, there must be a direct connection $L = l_1, \ldots, l_t$ from $V(Y)$ to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h)$. Assume w.l.o.g. that l_t is adjacent to $p \in V(\Gamma) \setminus \{h, i, j, k\}$. If

$p \in V(T) \cup V(Q) \cup V(R) \setminus \{h, k\}$, the nodes in $V(Y) \cup V(L)$ induce a direct connection, between $y \in N_{PS}(h)$ and $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$, that violates Lemma 7.5. If $p \in (V(P) \cup V(S)) \setminus \{h, i, j\}$, the nodes in $V(Y) \cup V(L)$ induce a path from p or from i or from j that violates Lemma 4.1. This completes the proof of the first part of the Corollary. The proof of the second part is identical. □

Corollary 7.8 *Suppose $y \in N_{PS}(h)$ and $w \in N_{QR}(h)$. Let $N(\Gamma)$ be the set of nodes adjacent to at least one node in Γ. Then there cannot be a direct connection between y and w avoiding $V(\Gamma) \cup N(\Gamma) \setminus \{y, w\}$.*

Proof: Suppose the contrary. Then there exists a direct connection $Y = y_1, \ldots, y_m$, between y and w avoiding $Z(h) \setminus \{y, w\}$. Since $Z(h)$ is an extended star but not a cutset, there must be a direct connection $L = l_1, \ldots, l_t$ from $V(Y)$ to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h)$. Assume w.l.o.g. that l_t is adjacent to $p \in V(\Gamma) \setminus \{h, i, j, k\}$. If $p \in V(T) \cup V(Q) \cup V(R) \setminus \{h, k\}$, the nodes in $V(Y) \cup V(L)$ induce a direct connection, between $y \in N_{PS}(h)$ and $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$, that violates Lemma 7.6. If $p \in (V(P) \cup V(S)) \setminus \{h, i, j\}$, the nodes in $V(Y) \cup V(L)$ induce a path from $w \in N_{QR}(h)$ to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{w\}$, that violates Lemma 7.6. □

Lemma 7.9 *Assume that h has at least one twin in Γ and let $h_1, \ldots, h_q$ be its twins. If the top path T of Γ has length greater than 1, then exactly one of the following holds:*

(i) $N_{PS}(h) = N_{PS}(h_t)$ *for* $t = 1, 2, \ldots, q$.

(ii) $N_{QR}(h) = N_{QR}(h_t)$ *for* $t = 1, 2, \ldots, q$.

Proof: We first prove that $N_{PS}(h) = N_{PS}(h_t)$ or $N_{QR}(h) = N_{QR}(h_t)$ holds for $t = 1, 2, \ldots, q$. Suppose the contrary. Then there exists a twin h_t, $1 \leq t \leq q$, such that $N_{PS}(h) \neq N_{PS}(h_t)$ and $N_{QR}(h) \neq N_{QR}(h_t)$.

Claim 1 *If $y \in N_{PS}(h)$ and $y \notin N_{PS}(h_t)$ then y is not a neighbor of h_t.*

Proof of Claim 1: By Definition 7.5, there exists a Type 1 or Type 2[7.1] direct connection $Y = y_1, \ldots, y_m$ from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$. Assume w.l.o.g. that y_1 and y are adjacent. Suppose y is a neighbor of h_t. Now no node of Y is adjacent to h_t, otherwise there is a wheel with center

h_t. So Y is of Type 1 or 2[7.1] in the goggles obtained from Γ by replacing h by h_t, contradicting the hypothesis that $y \notin N_{PS}(h_t)$. This completes the proof of Claim 1.

The proof of the following claim is identical to the previous one.

Claim 2 *If $y \in N_{QR}(h)$ and $y \notin N_{QR}(h_t)$ then y is not a neighbor of h_t.*

Claim 3 *There exist two nodes y and w satisfying one of the following properties:*

- *$y \in N_{PS}(h)$, $y \notin N_{PS}(h_t)$, $w \in N_{QR}(h_t)$ and $w \notin N_{QR}(h)$.*
- *$y \in N_{PS}(h_t)$, $y \notin N_{PS}(h)$, $w \in N_{QR}(h)$ and $w \notin N_{QR}(h_t)$.*

Proof of Claim 3: Assume w.l.o.g. that there exists a node, say d, in $N_{PS}(h)$ and $d \notin N_{PS}(h_t)$. Suppose there is no node which is in $N_{QR}(h_t)$ and not in $N_{QR}(h)$. Now since $N_{QR}(h) \neq N_{QR}(h_t)$, there must be a node, say $f \in N_{QR}(h)$ and $f \notin N_{QR}(h_t)$. Moreover $N_{QR}(h_t) \subset N_{QR}(h)$. Now there must be a node, say g, which is a neighbor of h_t and not a neighbor of h, otherwise $N(h) \cup \{h\}$ is a star cutset of G separating h_t from $V(G) \setminus \{h_t\}$. Now it follows that $g \in N_{PS}(h_t)$ and $g \notin N_{PS}(h)$. Nodes f and g prove Claim 3.

Let f and g be two nodes such that $f \in N_{QR}(h)$, and $f \notin N_{QR}(h_t)$, $g \in N_{PS}(h_t)$ and $g \notin N_{PS}(h)$.

By Definition 7.5, there exists a Type 3[7.1] direct connection $X = x_1, \ldots, x_n$ from f to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{f\}$. Let Γ^* denote the goggles obtained from Γ by replacing h with h_t. By Definition 7.5, there exists a Type 1 or Type 2[7.1] direct connection $Y = y_1, \ldots, y_m$ from g to $V(\Gamma^*) \setminus \{h_t, i, j, k\}$ avoiding $Z(h_t) \setminus \{g\}$. By Lemma 7.2, we can assume w.l.o.g. that the nodes i, j and k are not adjacent to $x_1, \ldots, x_{n-1}$ or $y_1, \ldots, y_{m-1}$. Furthermore, if Y is of Type 1[7.1], assume w.l.o.g. that y_m is adjacent to $V(S)$.

Suppose node y_m is adjacent to a node in X. Now there is a violation of Lemma 4.1 or Lemma 6.1 depending upon whether Y is a Type 1 or Type 2[7.1] direct connection. Hence node y_m is not adjacent to a node in X. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 7.6 irrespective of whether or not h and h_t have neighbors in Y and X respectively.

Case 1 No node of Y is adjacent to h and no node of X is adjacent to h_t.

Now there is a $3PC(a,i)$ irrespective of whether Y is a Type 1 or Type 2[7.1] direct connection.

Case 2 A node y_l of Y is adjacent to h but no node of X is adjacent to h_t.

Since $Z(h)$ is an extended star but not a cutset, there exists a direct connection $W = w_1, \dots, w_u$ from $\{g, y_1, \dots, y_{l-1}\}$ to $V(\Gamma) \setminus \{h,i,j,k\}$ avoiding $Z(h)$. W.l.o.g. assume w_s is adjacent to $V(\Gamma) \setminus \{h,i,j,k\}$. If no node of W is adjacent to h_t then, by Lemma 7.6 applied to node g, $V(Y) \cup V(W)$ induces a Type 1 or 2[7.1] direct connection and we are back in Case 1. Let w_r be the node of W with largest index which is adjacent to node h_t. If $w_{r+1}, \dots, w_u$ is a direct connection of Type 1 or 2[7.1] from node w_r then, replacing g by w_r, we are back to Case 1. If $w_{r+1}, \dots, w_u$ is a direct connection of Type 3[7.1] from node w_r then, replacing g by w_r and f by y_l, we are back to Case 1.

Case 3 A node of X is adjacent to h_t but no node of Y is adjacent to h.

The proof is analogous to Case 2.

Case 4 A node of Y is adjacent to h and a node of X is adjacent to h_t.

There is a $3PC(a,i)$.

Thus $N_{PS}(h) = N_{PS}(h_t)$ or $N_{QR}(h) = N_{QR}(h_t)$ holds for $t = 1, 2, \dots, q$.

Now repeating the same argument with h replaced by one of its twins, say h_s, it follows that $N_{PS}(h_s) = N_{PS}(h_t)$ or $N_{QR}(h_s) = N_{QR}(h_t)$ for every pair of twins h_s and h_t.

Now if the lemma is false we have that for some pair of twins s and t, $N_{PS}(h) \neq N_{PS}(h_t)$ and $N_{QR}(h) \neq N_{QR}(h_s)$. It follows that $N_{PS}(h) = N_{PS}(h_s) \neq N_{PS}(h_t)$ and $N_{QR}(h) = N_{QR}(h_t) \neq N_{QR}(h_s)$, a contradiction.

Now if both (i) and (ii) hold, $N(h) \cup \{h\}$ is a star cutset of G separating the twins of h, from the rest of the graph. This completes the proof of the lemma. □

Lemma 7.10 *Suppose $h_1, \dots, h_q$ are the twins of h in Γ. If the top path has length 1, then $N_{QR}(h) = N_{QR}(h_t)$ for $t = 1, \dots, q$.*

Proof: Suppose the lemma is false. Then there exists a twin h_t of h such that $N_{QR}(h) \neq N_{QR}(h_t)$. Assume w.l.o.g. that there exists a node y in $N_{QR}(h)$ but not in $N_{QR}(h_t)$. By Lemma 7.4 and Definition 7.4, there exists a Type 3[7.1] direct connection Y and a Type 2[7.1] direct connection X from y to $V(\Gamma) \setminus \{h, i, j, k\}$ avoiding $Z(h) \setminus \{y\}$. Moreover no node of Y is coincident with or adjacent to a node of X. Now consider the goggles Γ^* obtained from Γ by replacing h with h_t. No node of Y is adjacent to h_t, otherwise there is a Type 3[7.1] path that violates Lemma 7.3. Now y and h_t are not adjacent, otherwise $y \in N_{QR}(h_t)$. No node of X is adjacent to h_t, otherwise there is a Type 3[7.1] path that violates Lemma 7.3. Now the nodes in X and Y induce a direct connection from a Type a[2.1] node that violates Lemma 6.1.□

8 Direct Connections from a Node of Type b

Let $w \in V^r$ be a Type b[2.1] node adjacent to i and j. Let W be the set of Type b[2.1] nodes adjacent to i and j, but distinct from w. In the partial graph $G \setminus \{wi, wj\}$ let $X = x_1, \ldots, x_n$ be a direct connection between w and $V(\Gamma)$ avoiding W. Assume w.l.o.g. that x_1 is adjacent to w and x_n is adjacent to $t \in V(\Gamma)$.

Lemma 8.1 *In $G \setminus \{wi, wj\}$, every direct connection X between w and $V(\Gamma)$, avoiding W is one of the following types, see Figure 8.*

Type 1 *Node x_1 is adjacent to h but not strongly adjacent to Γ.*

Type 2 *Node x_n is not strongly adjacent to Γ and $t \in V^c \cap V(T)$ or $x_n \in V^c$ is a twin of a node in $V(T)$.*

Proof: There are two cases:

Case 1 Node x_n is not strongly adjacent to Γ.

If $t = h$, then $n = 1$, otherwise there is a violation of Lemma 4.1. This yields a path of Type 1. Suppose $t \neq h$. If $t \in V(P) \cup V(S)$, there is a violation of Lemma 4.1. If $t \in V(T) \cap V^c$, we have a path of Type 2. If $t \in V(T) \cap V^r$, there is a $3PC(i, t)$. If $t \in V(Q) \cup V(R) \setminus \{v\}$, there is a violation of Lemma 4.1.

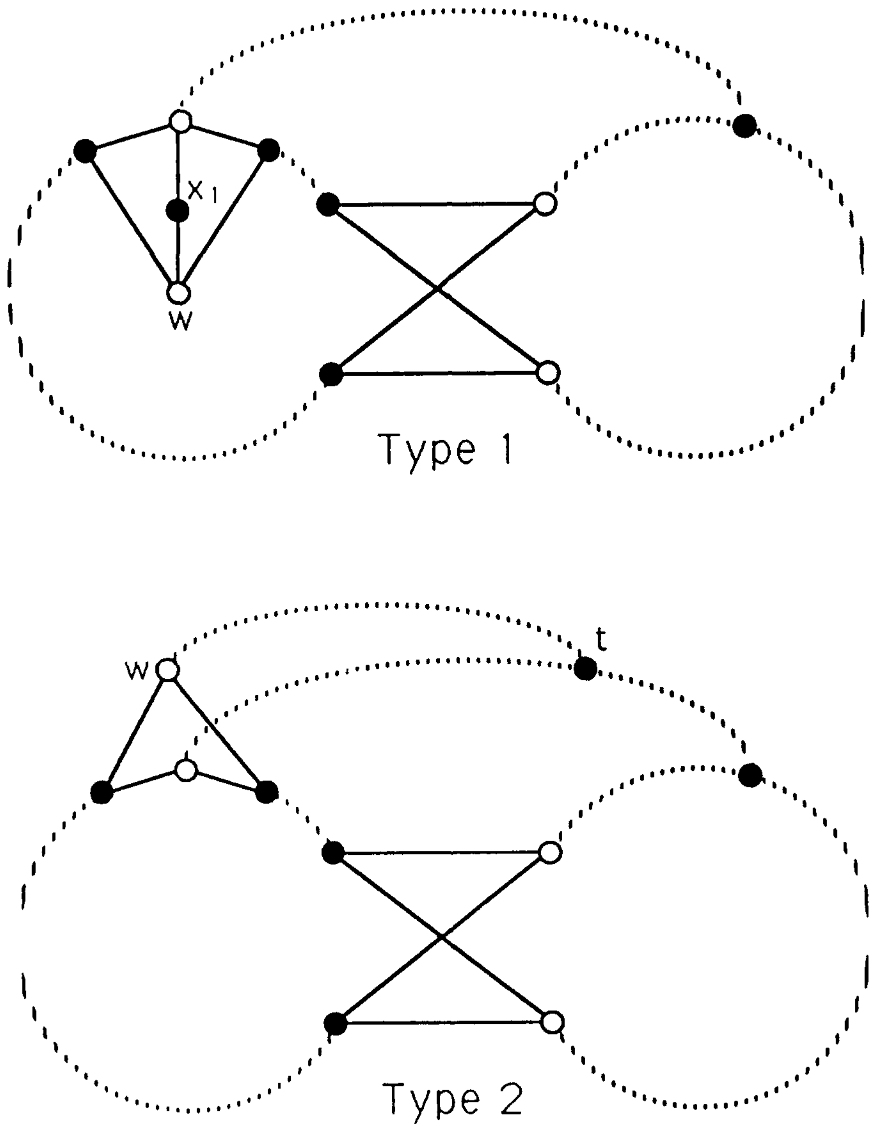


Figure 8: Direct Connections from a type b Node

Case 2 Node x_n is strongly adjacent to Γ.

Suppose x_n is a twin of a node d in Γ. Then modify Γ by replacing d with x_n and consider the direct connection $X^* = x_1, \ldots, x_{n-1}$. Note that $n \geq 2$, otherwise w is a strongly adjacent node that violates Theorem 2.2 with respect to the modified goggles. Now, with respect to the modified goggles, we are back to Case 1.

Suppose x_n is not a twin of a node in Γ.

If x_n is a Type a[2.1] node, there is a violation of Lemma 6.1. If x_n is a Type d[2.1] node, there is a violation of Lemma 5.1. If x_n is a Type b[2.1] node, adjacent to $q \in V(Q) \cap N(v)$ and $r \in V(R) \cap N(v)$, there is a $3PC(i, q)$. Suppose x_n is a Type c[2.1] node. Then, by Lemma 3.5, node x_n must be adjacent to $t \in V(T) \cap N(v)$ and to either $q \in V(Q) \cap N(v)$ or $r \in V(R) \cap N(v)$. Now there is a $3PC(i, t)$. $\square$

Corollary 8.2 *Suppose w is a Type b[2.1] node adjacent to i and j. Then there must exist a Type 2[8.1] direct connection $X = x_1 \ldots x_n$ between w and $V(\Gamma)$ such that x_n is not adjacent to k and x_n is not a twin of k.*

Proof: Apply Corollary 1.1 to the parachute with top path i, w, j, side paths i, P_{ix}, x, a and j, S_{ju}, u, a, center node h and middle path h, T, v, Q, a. Let $X = x_1 \ldots, x_n$ be the resulting path of Type d[3.3(III)] or d1[4.1(III)]. Assume w.l.o.g. that x_1 is adjacent to w. No node of X is adjacent to a node in R, otherwise there is a violation of Lemma 8.1. Now node x_n is not adjacent to k and x_n is not a twin of k. $\square$

Lemma 8.3 *Suppose $y \in N_{PS}(h)$ and there exists a Type b[2.1] node adjacent to i and j. Then node y is adjacent to all Type b[2.1] nodes that are adjacent to i and j. Moreover y is adjacent to all the twins of h.*

Proof: Suppose $y \in N_{PS}(h)$ is not adjacent to node w , a Type b[2.1] node adjacent to i and j. By Lemma 3.6, the top path T of Γ is of length greater than one. Let $X = x_1, \ldots, x_m$ be a Type 2[8.1] direct connection between w and $V(\Gamma) \setminus \{h, i, j, k\}$. Assume w.l.o.g. that x_m is adjacent to $t \in V(T) \setminus \{h, k\}$. Note that by Corollary 8.2 such a direct connection must exist. No node of X is adjacent to h^*, a twin of h, otherwise there is a wheel with h^* as center. Let $Z(h)$ comprise:

- The set of nodes h, i, j, k and their twins relative to Γ.
- The nodes of $N(h)$.
- The Type b[2.1] nodes adjacent to i and j.

Note that, by Lemma 3.5, there cannot be a strongly adjacent node of Type c[2.1] adjacent to k and either i or j.

Now by Definition 7.5, there must be a Type 1 or Type 2[7.1] direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$, avoiding $Z(h) \setminus \{y\}$.

There are two cases:

Case 1 Direct connection Y is of Type 1[7.1].

Assume w.l.o.g. that y_n is adjacent to $p \in V(S) \setminus \{h, j\}$. Nodes y_n and x_m are not adjacent, otherwise there is a violation of Lemma 4.1. By Lemma 7.1, no node of Y is adjacent to k or i. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 7.6 or Lemma 8.1. Now there is a $3PC(a, i)$. Hence w and y are adjacent.

Suppose now y is not adjacent to h^*, a twin of h. No node of Y is adjacent to h^*, otherwise there is a $3PC(a, i)$. Now there is a $3PC(w, x_m)$ or a $3PC(w, t)$ depending on whether x_m is a twin of a node in T or not. Hence y and h^* are adjacent.

Case 2 Direct connection Y is of Type 2[7.1].

Assume w.l.o.g. that y_n is a Type a[2.1] node.

By Lemma 7.1, y_n is adjacent to a and b and no node of Y is adjacent to i,j or k. Nodes y_n and x_m are not adjacent, otherwise there is a violation of Lemma 6.1. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 7.6 or Lemma 8.1. Now there is a $3PC(a, i)$. Hence w and y are adjacent.

Suppose now y is not adjacent to h^*, a twin of h. No node of Y is adjacent to h^*, otherwise there is a $3PC(a, i)$. Now there is a $3PC(w, x_m)$ or a $3PC(w, t)$ depending on whether x_m is a twin of a node in T or not. Hence y and h^* are adjacent.

This completes the proof of the lemma. □

Remark 8.4 *Suppose there exists a Type b[2.1] node adjacent to i and j. Let $h_1 \ldots, h_q$ be the twins of h. Then by Lemma 8.3, $N_{PS}(h) = N_{PS}(h_t)$ for $t = 1, 2, \ldots, q$.*

9 Direct Connections from a Node of Type c

Let $w \in V^r$ be a Type c[2.1] node adjacent to k and adjacent to either i or j. Assume w.l.o.g. that w is adjacent to i. Let W be the set of Type c[2.1] nodes adjacent to k and i but distinct from w. In the partial graph $G \setminus \{wk, wi\}$, let $X = x_1, \ldots, x_n$ be a direct connection between w and $V(\Gamma)$ avoiding W. Assume w.l.o.g. that x_1 is adjacent to w and x_n is adjacent to $t \in V(\Gamma)$.

Lemma 9.1 *In $G \setminus \{wk, wi\}$, every direct connection X between w and $V(\Gamma)$, avoiding W is one of the following types, see Figure 9.*

Type 1 *Node x_1 is adjacent to h but not strongly adjacent to Γ.*

Type 2 *Node x_n is not strongly adjacent to Γ and $t \in V^c \cap V(S)$ or $x_n \in V^c$ is a twin of a node in $V(S)$.*

Type 3 *Node x_1 is a Type c[2.1] node. The top path T of Γ is of length 1. Node x_1 is adjacent to h and to the node in $V(Q) \cap N(v)$ or to the node in $V(R) \cap N(v)$.*

Proof: There are two cases:

Case 1 Node x_n is not strongly adjacent to Γ.

If $t = h$, then $n = 1$, otherwise there is a violation of Lemma 4.1. This yields a path of Type 1. Suppose $t \neq h$. If $t \in V(Q) \cup V(R)$, there is a violation of Lemma 4.1. If $t \in V(S)$ then $t \in V^c$ and we have a path of Type 2, otherwise there is a violation of Lemma 4.1. If $t \in V(P)$ there is a violation of Lemma 4.1. If $t \in V(T) \setminus \{h, k\}$, there is a $3PC(a, i)$. By Lemma 1.2, $t \neq k$.

Case 2 Node x_n is strongly adjacent to Γ.

Suppose x_n is a twin of a node, say d, in Γ. Then modify Γ by replacing d with x_n and consider the direct connection $X^* = x_1, \ldots, x_{n-1}$.

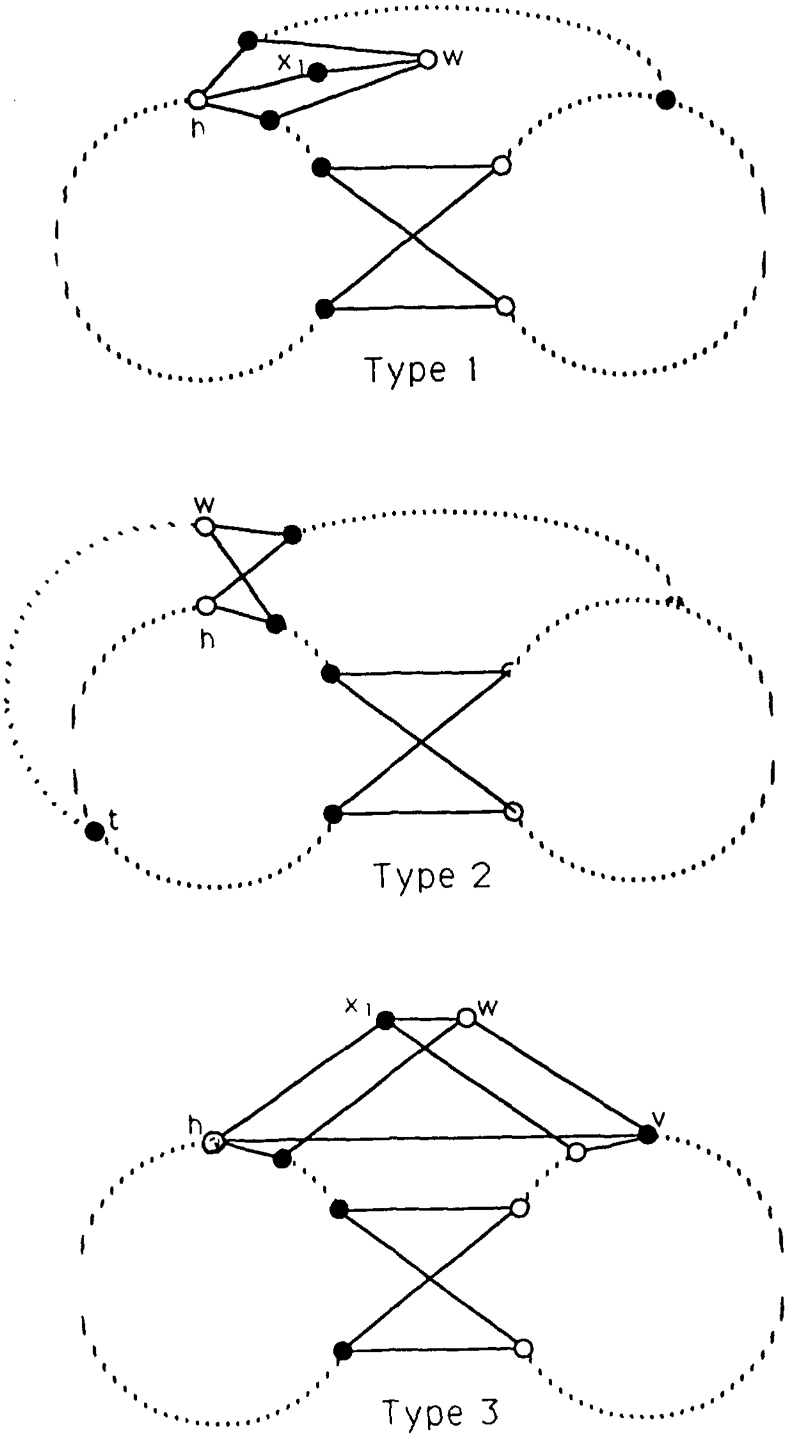


Figure 9: Direct Connections from a Type c Node

Note that $n \geq 2$, otherwise w is a strongly adjacent node that violates Theorem 2.2 with respect to the modified goggles. Now, with respect to the modified goggles, we are back to Case 1.

Suppose x_n is not a twin of a node in Γ. If x_n is a Type a[2.1] node, there is a violation of Lemma 6.1. If x_n is a Type d[2.1] node, there is a violation of Lemma 5.1. If x_n is a Type b[2.1] node then, by Lemma 3.5, node x_n is adjacent to $q \in V(Q) \cap N(v)$ and $r \in V(R) \cap N(v)$. Now there is a violation of Lemma 8.1. Suppose x_n is a Type c[2.1] node. Then, by Lemma 3.5, node x_n must be adjacent to $t \in V(T) \cap N(v)$ and to either $q \in V(Q) \cap N(v)$ or $r \in V(R) \cap N(v)$. Assume w.l.o.g. that x_n is adjacent to q. If the top path T of Γ is of length greater than 1 there is a $3PC(x, q)$. Suppose the top path T of Γ is of length 1. Now, by Lemma 3.11, w and x_n are adjacent. Hence $n = 1$ and we have a path of Type 3. □

Corollary 9.2 *Suppose w is a Type c[2.1] node adjacent to k and i. Then there must exist a Type 2[9.1] direct connection $X = x_1 \dots x_n$ between w and $V(\Gamma)$ such that x_n is not adjacent to j and x_n is not a twin of j.*

Proof: Apply Corollary 1.1 to the parachute with top path k, w, i, side paths k, T_{kv}, v, Q, a and i, P_{ix}, x, a, center node h and middle path h, S, u, a. Let $X = x_1 \dots, x_n$ be the resulting path of Type d[3.3(III)] or d1[4.1(III)]. Assume w.l.o.g. that x_1 is adjacent to w. No node of X is adjacent to a node in R, otherwise there is a violation of Lemma 9.1. Now node x_n is not adjacent to j and x_n is not a twin of j. □

Lemma 9.3 *Suppose $y \in N_{QR}(h)$ and a Type c[2.1] node that is adjacent to k and i exists. Then y is adjacent to all Type c[2.1] nodes that are adjacent to i and k. Moreover y is adjacent to all the twins of h.*

Proof: Suppose $y \in N_{QR}(h)$ is not adjacent to w , a Type c[2.1] node adjacent to k and i. Let $X = x_1, \dots, x_m$ be a Type 2[9.1] direct connection between w and $V(\Gamma) \setminus \{h, i, j, k\}$. Assume w.l.o.g. that x_m is adjacent to $t \in V(\Gamma) \setminus \{h, i, j, k\}$. Note that by Corollary 9.2 such a direct connection must exist. No node of X is adjacent to h^*, a twin of h, otherwise there is a wheel with h^* as center. Let $Z(h)$ comprise:

- The set of nodes h, i, j, k and their twins relative to Γ.

- The nodes of $N(h)$.
- The Type c[2.1] nodes adjacent to k and i.

Note that, by Lemma 3.5, there cannot be a Type c[2.1] node adjacent to k and j and there cannot be a Type b[2.1] node adjacent to i and j. There are two cases.

Case 1 The top path T is of length greater than 1.

Now by Definition 7.5, there must be a Type 3[7.1] direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$, avoiding $Z(h) \setminus \{y\}$. There are two subcases.

Case 1.1 Node y_n is adjacent to $p \in V(T) \setminus \{h, k\}$.

Nodes y_n and x_m are not adjacent, otherwise there is a violation of Lemma 4.1. By Lemma 7.1, no node of Y is adjacent to i or j. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 7.6 or Lemma 9.1. Now there is a $3PC(a, i)$. Hence w and y are adjacent.

Suppose now y is not adjacent to h^*, a twin of h. No node of Y is adjacent to h^*, otherwise there is a $3PC(a, i)$. Now there is a $3PC(w, x_m)$ or a $3PC(w, t)$ depending on whether x_m is a twin of a node in S or not. Hence y and h^* are adjacent.

Case 1.2 Node y_n is a Type b[2.1] node.

By Lemma 7.1, y_n is adjacent to $q \in V(Q) \cap N(v)$ and $r \in V(R) \cap N(v)$ and no node of Y is adjacent to i, j or k. Nodes y_n and x_m are not adjacent, otherwise there is a violation of Lemma 8.1. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 7.6 or Lemma 9.1. Now there is a $3PC(a, i)$. Hence w and y are adjacent.

Suppose y is not adjacent to h^*, a twin of h. No node of Y is adjacent to h^*, otherwise there is a $3PC(a, i)$. Now there is a $3PC(w, x_m)$ or a $3PC(w, t)$ depending on whether X_m is a twin of a node in S or not. Hence y and h^* are adjacent.

Case 2 The top path is of length 1.

Now by Definition 7.5, there must be a Type 3[7.3] direct connection $Y = y_1, \ldots, y_n$ between y and $V(\Gamma) \setminus \{h, i, j, k\}$, avoiding $Z(h) \setminus \{y\}$. Assume w.l.o.g that y_1 is adjacent to y. By Lemma 7.3, y_n is adjacent to x and u and no node of Y is adjacent to i or j. Moreover, y_1 is the unique node of Y adjacent to v. Nodes x_m and y_n are not adjacent, otherwise there is a violation of Lemma 6.1. No node of Y is adjacent to a node of X, otherwise there is a violation of Lemma 6.1 or Lemma 4.1. Now there is a violation of Lemma 1.3. Hence w and y are adjacent. By Lemma 7.10, y is adjacent to all the twins of h. This completes the proof of the lemma. □

Remark 9.4 *Suppose there exists a Type c[2.1] node adjacent to k and i. Let $h_1 \ldots, h_q$ be the twins of h. Then by Lemma 9.3, $N_{QR}(h) = N_{QR}(h_t)$ for $t = 1, 2, \ldots, q$.*

10 2-Join Theorem

In this final section of the paper we prove a 2-join theorem.

Theorem 10.1 *Suppose G is a bipartite graph that is signable to be balanced, contains goggles and does not contain a wheel, connected squares, a connected 6-hole, an R_{10} configuration or an extended star cutset. Then G contains goggles $Go(P, Q, R, S, T)$ and a 2-join separating $V(P) \cup V(S) \setminus \{h\}$ from $V(Q) \cup V(R) \cup V(T) \setminus \{h\}$.*

Proof: Among the goggles of G, let Γ be one with shortest top path T and, subject to this condition, with the fewest number of nodes. There are three cases depending upon whether the top path T is of length 1 or of length greater than 1 and whether $N_{PS}(h) = N_{PS}(h_t)$ for $t = 1, 2, \ldots, q$ or $N_{QR}(h) = N_{QR}(h_t)$ for $t = 1, 2, \ldots, q$, where $h_1, \ldots, h_q$ are the twins of h.

Case 1 The top path T of Γ has length greater than 1 and if a twin of h exists, $N_{PS}(h) = N_{PS}(h_t)$ for $t = 1, 2, \ldots, q$ where $h_1, \ldots, h_q$ are the twins of h.

Now a Type c[2.1] node adjacent to k and either i or j cannot exist by Lemma 7.9 and Remark 9.4. A Type b[2.1] node adjacent to i and j may exist. The nodes of Γ and Type a, b, c, d[2.1] nodes relative to Γ are partitioned into six sets as follows:

- The set A comprising of nodes x, u and their twins and Type a[2.1] nodes adjacent to a and b.
- The set B comprising of nodes a, b and their twins and Type a[2.1] nodes adjacent to u and x.
- The set D comprising of nodes i, j and their twins and nodes in $N_{PS}(h)$.
- The set F comprising of nodes h, and its twins and Type b[2.1] nodes adjacent to i and j.
- The set M comprising of Type d[2.1] nodes in V^c and the nodes in $V(P) \cup V(S) \setminus \{h, i, j, u, x\}$.
- The set N comprising of Type b[2.1] nodes adjacent to a node in $V(Q) \cap N(v)$ and a node in $V(R) \cap N(v)$, Type c[2.1] nodes adjacent to a node in $V(T)$ and a node in $V(Q) \cup V(R)$, Type d[2.1] nodes in V^r and nodes in $V(Q) \cup V(R) \cup V(T) \setminus \{h, a, b\}$.

By Remark 3.4 and Lemma 6.3, the nodes in $A \cup B$ induce a biclique K_{AB}. By Remarks 3.9 and 8.4, the nodes in $D \cup F$ induce a biclique K_{DF}. We now prove that the edge set $E^* = E(K_{AB}) \cup E(K_{DF})$ is a 2-join of the graph G. Suppose not. Then in the partial graph $G \setminus E^*$ there must be a direct connection $Y = y_1 \ldots, y_m$ between $A \cup D \cup M$ and $B \cup F \cup N$. Assume w.l.o.g. that y_1 is adjacent to a node in $A \cup D \cup M$ and y_m is adjacent to a node in $B \cup F \cup N$. Note that $m > 1$, otherwise there is a violation of Theorem 2.2, Lemma 5.1, Lemma 6.1, Lemma 7.1, Lemma 7.6 or Lemma 8.1.

Suppose y_1 is a twin of a node in Γ, say d. Then for the goggles Γ^* obtained from Γ by replacing d with y_1 consider the direct connection $Y^* = y_2 \ldots, y_m$. Note that by Remarks 3.11 and 8.4, the sets of nodes A^*, B^*, D^* and F^* relative to Γ^* are equal to the sets of nodes A, B, D F respectively. Hence the edges in the 2-join are the same and the partial graph $G \setminus E^*$ remains the same. Now, it follows that $m > 2$. So assume w.l.o.g. that y_1 is not a twin of a node in Γ. By repeating the same argument with y_m, we assume w.l.o.g. that y_m is also not a twin of a node in Γ. There are three subcases. In each subcase there is a violation of one of the previous lemmas. Note that y_1 may be adjacent to a node of Γ and to a Type a, b or d[2.1] node. If so the violation is with respect to the direct connection in which y_1 is adjacent to a node of Γ. Similarly y_m may be adjacent to a node of Γ and to a Type a, b, c, or d[2.1] node. If so the violation is with respect to the direct connection in which y_m is adjacent to a node of Γ.

Case 1.1 Node y_1 is adjacent to a node in A.

Suppose y_1 is adjacent to x or u or a twin of x or a twin of u. If y_m is adjacent to h then, since edge $y_m h \notin E^*$, node y_m is in $N_{QR}(h)$ and there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 4.1.

Suppose y_1 is adjacent to a Type a[2.1] node adjacent to a and b. If y_m is adjacent to a Type a[2.1] node adjacent to u and x, there is a violation of Lemma 6.5. If y_m is adjacent to h then y_m is in $N_{QR}(h)$ and there is a violation of Lemma 7.7. Otherwise there is a violation of Lemma 6.1.

Case 1.2 Node y_1 is adjacent to a node in D.

Suppose y_1 is adjacent to i or j or a twin of i or a twin of j. Now y_m must be adjacent to h, otherwise there is a violation of Lemma 4.1. But then y_m is in $N_{QR}(h)$ and there is a violation of Lemma 7.6. If y_1 is adjacent to a node in $N_{PS}(h)$, there is a violation of Lemma 7.6 or Corollary 7.8.

Case 1.3 Node y_1 is adjacent to a node in M.

Suppose y_1 is adjacent to a node in $V(P) \cup V(S) \setminus \{h, i, j, u, x\}$. If y_m is adjacent to h then y_m is in $N_{QR}(h)$ and there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 4.1. If y_1 is adjacent to a Type d[2.1] node, there is a violation of Lemma 5.1.

Case 2 The top path T of Γ has length greater than 1 and if a twin of h exists, $N_{QR}(h) = N_{QR}(h_t)$ for $t = 1, 2, \ldots, q$ where $h_1, \ldots, h_q$ are the twins of h. Now a Type b[2.1] node adjacent to i and j cannot exist by Lemma 7.9 and Remark 8.4. A Type c[2.1] node adjacent to k and either i or j may exist. The nodes of Γ and Type a, c, d[2.1] nodes relative to Γ are partitioned into six sets as follows:

- The set A comprising of nodes x, u and their twins and Type a[2.1] nodes adjacent to a and b.
- The set B comprising of nodes a, b and their twins and Type a[2.1] nodes adjacent to u and x.
- The set D comprising of nodes h and its twins and Type c[2.1] nodes adjacent to k and i.
- The set F comprising of nodes k and its twins and nodes in $N_{QR}(h)$.
- The set M comprising of Type c[2.1] nodes adjacent to k and either i or j, Type d[2.1] nodes in V^c and the nodes in $V(P) \cup V(S) \setminus \{h, u, x\}$.
- The set N comprising of Type b[2.1] nodes adjacent to a node in $V(Q) \cap N(v)$ and a node in $V(R) \cap N(v)$, Type c[2.1] nodes adjacent to a node in $V(T)$ and a node in $V(Q) \cup V(R)$, Type d[2.1] nodes in V^r and nodes in $V(Q) \cup V(R) \cup V(T) \setminus \{h, a, b\}$.

By Remark 3.4 and Lemma 6.3,the nodes in $A \cup B$ induce a biclique K_{AB}. By Remarks 3.9 and 9.4,the nodes in $D \cup F$ induce a biclique K_{DF}. We now prove that the edge set $E^* = E(K_{AB}) \cup E(K_{DF})$ is a 2-join of the graph G.

Suppose not. Then in the partial graph $G \setminus E^*$ there must be a direct connection $Y = y_1, \ldots, y_m$ between $A \cup D \cup M$ and $B \cup F \cup N$. Assume w.l.o.g. that y_1 is adjacent to a node in $A \cup D \cup M$ and y_m is adjacent to a node in $B \cup F \cup N$. Note that $m > 1$, otherwise there is a violation of Theorem 2.2. By the same arguments used in Case 1, we assume w.l.o.g. that both y_1 and y_m are not twins of nodes in Γ. There are three subcases:

Case 2.1 Node y_1 is adjacent to a node in A.

Suppose y_1 is adjacent to x or u or a twin of x or a twin of u. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 4.1. Suppose y_1 is adjacent to a Type a[2.1] node adjacent to a and b. If y_m is adjacent to a Type a[2.1] adjacent to u and x, there is a violation of Lemma 6.5. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 6.1.

Case 2.2 Node y_1 is adjacent to a node in D.

Suppose y_1 is adjacent to h. Then $y_1 \in N_{PS}(h)$. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Corollary 7.8, otherwise there is a violation of Lemma 7.6. If y_1 is adjacent to a Type c[2.1] node which is adjacent to i and k, there is a violation of Lemma 9.1.

Case 2.3 Node y_1 is adjacent to a node in M.

Suppose y_1 is adjacent to a node in $V(P) \cup V(S) \setminus \{h, u, x\}$. If y_m is adjacent to a node in $N_{QR}(H)$ there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 4.1. If y_1 is adjacent to a Type c[2.1] node, there is a violation of Lemma 9.1. If y_1 is adjacent to a Type d[2.1] node, there is a violation of Lemma 5.1.

Case 3 The top path T of Γ is of length 1.

By Lemma 3.6, there are no Type b[2.1] nodes. Assume w.l.o.g. that a Type c[2.1] node, if it exists, is adjacent to v and i. By Lemma 3.5, there is no Type c[2.1] node adjacent to v and j. Let $U_{ab} = \{w |\ w$ is a Type a[2.1] node adjacent to a and $b\}$. Similarly, let $U_{ux} = \{w |\ w$ is a Type a[2.1] node adjacent to u and $x\}$. Let U_1 be the nodes in U_{ab} that are adjacent to all nodes in U_{ux} and let $U_2 = U_{ab} \setminus U_1$. The nodes of Γ and Type a, c, d[2.1] nodes relative to Γ are partitioned into six sets as follows:

- The set A comprising of nodes x, u and their twins and Type a[2.1] nodes in U_1.
- The set B comprising of nodes a, b and their twins and Type a[2.1] nodes in U_{ux}.
- The set D comprising of nodes h and its twins and Type c[2.1] nodes adjacent to v and i.
- The set F comprising of nodes v and its twins, Type c[2.1] nodes adjacent to h and nodes in $N_{QR}(h)$.
- The set M comprising of Type d[2.1] nodes in V^c and the nodes in $V(P) \cup V(S) \setminus \{h, u, x\}$.
- The set N comprising of Type d[2.1] nodes in V^r, nodes in U_2 and the nodes in $V(Q) \cup V(R) \setminus \{v, a, b\}$.

By Remark 3.4 and the definition of the sets U_1 and U_2, the nodes in $A \cup B$ induce a biclique K_{AB}. By Remarks 3.13 and 9.4, the nodes in $D \cup F$ induce a biclique K_{DF}. We now prove that the edge set $E^* = E(K_{AB}) \cup E(K_{DF})$ is a 2-join of the graph G.

Suppose not. Then in the partial graph $G \setminus E^*$ there must be a direct connection $Y = y_1, \ldots, y_m$ between $A \cup D \cup M$ and $B \cup F \cup N$. Assume w.l.o.g. that y_1 is adjacent to a node in $A \cup D \cup M$ and y_m is adjacent to a node in $B \cup F \cup N$. Note that $m > 1$, otherwise there is a violation of Theorem 2.2. By the same arguments used in Case 1, we assume w.l.o.g. that both y_1 and y_m are not twins of nodes in Γ. There are three subcases:

Case 3.1 Node y_1 is adjacent to a node in A.

Suppose y_1 is adjacent to x or u or a twin of x or a twin of u. If y_m is adjacent to a node in U_2 there is a violation of Lemma 6.4. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 4.1.

Suppose y_1 is adjacent to a Type a[2.1] node $w \in U_1$. If y_m is adjacent to a Type a[2.1] node $y \in U_{ux}$, by the definition of the sets U_1, U_{ux}, A and B, it follows that $wy \in E^*$ and there is a violation of Lemma 6.5. If y_m is adjacent to a Type a[2.1] node $y \in U_2$, there is a violation of Lemma 6.6. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Lemma 7.6. Otherwise there is a violation of Lemma 6.1.

Case 3.2 Node y_1 is adjacent to a node in D.

If y_1 is adjacent to h then $y_1 \in N_{PS}(h)$. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Corollary 7.8. Otherwise there is a violation of Lemma 7.6. If y_1 is adjacent to Type c[2.1] node which is adjacent to v and [illegible], there is a violation of Lemma 9.1.

Case 3.3 Node y_1 is adjacent to a node in M.

Suppose y_1 is adjacent to a node in $V(P) \cup V(S) \setminus \{h, u, x\}$. If y_m is adjacent to a node in U_2 there is a violation of Lemma 6.3. If y_m is adjacent to a node in $N_{QR}(h)$ there is a violation of Lemma 7.6. Otherwise, there is a violation of Lemma 4.1. If y_1 is adjacent to a Type d[2.1] node, there is a violation of Lemma 5.1. $\square$

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1, K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1, K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T, A, N such that $T \neq \emptyset$, some node $x \varepsilon T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$,then $|A| \geq 2$ and every node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r, V^c; E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.